THE CURVE AND p-ADIC HODGE THEORY

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Contents

Abstract. The main theme of this course will be to understand and give a meaning to the notion of a p-adic Hodge structure. Starting with the work of Fontaine, who introduced many of the basic notions in the domain, it took many years to understand the exact definition of a p-adic Hodge structure. We now have the right definition: this involves the fundamental curve of p-adic Hodge theory and vector bundles on it. In the course I will explain the construction and basic properties of the curve. I will moreover explain the proof of the classification of vector bundles theorem on the curve. As an application I will explain the proof of weakly admissible implies admissible. In the meanwhile I will review many objects that show up in p-adic Hodge theory like p-divisible groups and their moduli spaces, Hodge-Tate and de Rham period morphisms, and filtered φ-modules.

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1. Introduction

1.1. **What is a p-adic Hodge structure?** Recall a *real pure Hodge structure* of weight $w \in \mathbb{Z}$ is a finitely dimensional real vector space V, endowed with a bigrading

$$V_{\mathbb{C}} = \bigoplus_{p+q=w} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$. For example, let $X/\mathbb{C}$ be a proper smooth algebraic variety. Then $\mathrm{H}^i(X(\mathbb{C}), \mathbb{R})$ is equipped with a real Hodge structure of weight i as

$$\mathrm{H}^i(X(\mathbb{C}), \mathbb{R})_{\mathbb{C}} = \bigoplus_{p+q=i} \mathrm{H}^q(X, \Omega^p).$$

In p-adic setting, there are plenty of different structures and results

- Hodge-Tate Galois representations;
- crystalline representations;
- de Rham representations;
- filtered φ-modules à la Fontaine;
- Breuil-Kisin modules;
- (φ, Γ)-modules;
- comparison theorems for proper smooth algebraic variety over $\mathbb{Q}_p$.

This is a mess! We should back to real case to find the solution.

1.2. **Real Hodge structure.** Recall Simpson's geometric point of view of twists. Denote

$$\widetilde{\mathbb{P}}^1_{\mathbb{R}} = \mathbb{P}^1_{\mathbb{C}} / \left\{ z \sim -\frac{1}{\bar{z}} \right\}$$

where z is the coordinate on $\mathbb{P}^1_{\mathbb{C}}$. This is a conic curve without real point, equipped with ∞. Obviouly $\mathbb{P}^1_{\mathbb{C}}$ is a double cover of $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$.

$$\begin{tikzcd} \mathbb{P}^1_{\mathbb{C}} \arrow[d, dash, "\pi", "\mathbb{Z}/2\mathbb{Z}"'] \\ \widetilde{\mathbb{P}}^1_{\mathbb{R}} \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[dr, dash] & & \infty \arrow[dl, dash] \\ & \infty & \end{tikzcd}$$

The action of $\mathbb{C}^\times$ on $\mathbb{P}^1_{\mathbb{C}}$ as $\lambda.z = \lambda z$ descends to an action of $U(1)$ on $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$. Then ∞ is the unique fixed point of this action and the unqiue point that has a finite orbit.

Consider the vector bundles on $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$. For $\lambda \in \frac{1}{2}\mathbb{Z}$, define

$$\mathcal{O}_{\widetilde{\mathbb{P}}^1_{\mathbb{R}}}(\lambda) = \begin{cases} \pi_* \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2\lambda), & \lambda \notin \mathbb{Z}; \\ \mathcal{L} \text{ such that } \pi^* \mathcal{L} = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2\lambda), & \lambda \in \mathbb{Z}. \end{cases}$$

Here the *slope* of $\mathcal{O}_{\widetilde{\mathbb{P}}^1_{\mathbb{R}}}(\lambda)$ is λ.

Proposition 1.1. *There is a bijection between the set of finite decreasing half integer sequences*

$$\left\{ \lambda_1 \geq \cdots \geq \lambda_n \mid \lambda_i \in \frac{1}{2}\mathbb{Z}, n \in \mathbb{N} \right\}$$

and the isomorphic classes of vector bundles on $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$ as

$$(\lambda_i) \longmapsto \left[\bigoplus_i \mathcal{O}_{\widetilde{\mathbb{P}}^1_{\mathbb{R}}}(\lambda_i) \right].$$

In particular,

$$\begin{aligned}\mathsf{Vect}_{\mathbb{R}} &\xrightarrow{\sim} \left\{\textit{slope } 0 \textit{ semisimple vector bundles over } \widetilde{\mathbb{P}}^1_{\mathbb{R}}\right\}\\ V &\longmapsto V \otimes \mathcal{O}_{\widetilde{\mathbb{P}}^1_{\mathbb{R}}}\\ \mathrm{H}^0(\widetilde{\mathbb{P}}^1_{\mathbb{R}}, \mathcal{E}) &\longleftarrow \mathcal{E}.\end{aligned}$$

That is to say, every Harder-Narasimhan filtration of vector bundles are split and every semisimple vector bundle of pure slope are $\mathcal{O}_{\widetilde{\mathbb{P}}^1_{\mathbb{R}}}(\lambda)^n$.

Let V be a real vector space with a filtration $\mathrm{Fil}^\bullet$ on $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. Denote by t the uniformization of $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$ at ∞ and

$$V_{\mathbb{C}}((t)) = V \otimes_{\mathbb{R}} \mathbb{C}((t)) = V_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{C}((t)).$$

There is a canonical filtration $\left\{t^k \mathbb{C}[[t]]\right\}_k$ on $\mathbb{C}((t))$, which induces a filtration on $V_{\mathbb{C}}((t))$ as

$$\mathrm{Fil}^k(V_{\mathbb{C}}((t))) = \sum_{i\in\mathbb{Z}} \mathrm{Fil}^i V_{\mathbb{C}} \otimes_{\mathbb{C}} t^{k-i}\mathbb{C}[[t]].$$

Then

$$\widehat{\mathcal{O}}_{\widetilde{\mathbb{P}}^1_{\mathbb{R}},\infty} = \mathbb{C}[[t]], \quad (V \otimes_{\mathbb{R}} \mathcal{O}_{\widetilde{\mathbb{P}}^1_{\mathbb{R}}})^\wedge_\infty = V_{\mathbb{C}}((t))$$

and the $\mathbb{C}[[t]]$-lattice

$$\Lambda := \mathrm{Fil}^0(V_{\mathbb{C}}((t))) \subset V_{\mathbb{C}}((t))$$

defines a *modification* of vector bundles

$$(V \otimes_{\mathbb{R}} \mathcal{O}_{\widetilde{\mathbb{P}}^1_{\mathbb{R}}})|_{\widetilde{\mathbb{P}}^1_{\mathbb{R}}\setminus\{\infty\}} \xrightarrow{\sim} \mathcal{E}|_{\widetilde{\mathbb{P}}^1_{\mathbb{R}}\setminus\{\infty\}},$$

such that $\widehat{\mathcal{E}}_\infty = \Lambda$. This is $U(1)$-equivalent and induces a bijection

$$\{\text{filtrations on } V_{\mathbb{C}}\} \xrightarrow{\sim} \left\{U(1)\text{-equiv. modif. } V \otimes_{\mathbb{R}} \mathcal{O}_{\widetilde{\mathbb{P}}^1_{\mathbb{R}}} \rightsquigarrow \mathcal{E}\right\}$$

and thus

$$\{(V, \mathrm{Fil}^\bullet V_{\mathbb{C}})\} \xrightarrow{\sim} \left\{\begin{array}{c} U(1)\text{-equiv. modif. } \mathcal{E}_1 \rightsquigarrow \mathcal{E}_2 \\ \mathcal{E}_1 \text{ semisimple of slope } 0,\ U(1) \curvearrowright \mathrm{H}^0(\mathcal{E}_1) \text{ trivially}\end{array}\right\}.$$

Definition 1.2. A *real Hodge structure* is a finitely dimensional real vector space V, endowed with a bigrading decomposition

$$V_{\mathbb{C}} = \bigoplus_{p,q\in\mathbb{Z}} V_{\mathbb{C}}^{p,q},$$

such that $\overline{V^{p,q}} = V^{q,p}$. Thus for any integer w, there is a subspace $V_w \subset V$ such that

$$V_{w,\mathbb{C}} = \bigoplus_{p+q=w} V^{p,q},$$

which is called *weight* w part of V. If $V = V_w$, V is called *pure of weight* w.

We say $(V, \mathrm{Fil}^\bullet V_{\mathbb{C}})$ defines a Hodge struture of weight w if there is a real Hodge struture on V of pure weight w such that $\mathrm{Fil}^n \mathbb{V}_{\mathbb{C}} = \oplus_{p\geq n} V^{p,w-p}$.

Proposition 1.3. $(V, \mathrm{Fil}^\bullet V_{\mathbb{C}})$ *defines a weight w Hodge struture if and only if $\mathcal{E}_2$ is semisimple of slope $w/2$ in the corresponding modification.*

This induces a bijection between the set of weight w pure real Hodge structures and the set of $U(1)$-equivalent modifications $\mathcal{E}_1 \rightsquigarrow \mathcal{E}_2$ on $\widetilde{\mathbb{P}}^1_{\mathbb{R}}\setminus\{\infty\}$, where $\mathcal{E}_1$ is semisimple of slope 0, $\mathcal{E}_2$ is semisimple of slope $w/2$ and $U(1)$ acts on $\mathrm{H}^0(\mathcal{E}_1)$ trivially.

WE are going to do the same in the p-adic setting.

real setting	p-adic setting
$\widetilde{\mathbb{P}}^1_{\mathbb{R}} \setminus \{\infty\} \curvearrowleft U(1)$	the curve $X \curvearrowleft \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$
$\mathbb{C}[[t]] = \widehat{\mathcal{O}}_{\widetilde{\mathbb{P}}^1_{\mathbb{R}}}$	$B^+_{\mathrm{dR}} = \widehat{\mathcal{O}}_{X,\infty}$
$\lambda.t = \lambda t$	$\sigma.t = \chi_{\mathrm{cyc}}(\sigma)t,\ t = \log[\epsilon]$
$\mathbb{P}^1_{\mathbb{C}} \xrightarrow{\mathbb{Z}/2\mathbb{Z}} \widetilde{\mathbb{P}}^1_{\mathbb{R}}$	$X_\infty \xrightarrow{\widehat{\mathbb{Z}}} X$

Thus the vector bundles on X is endowed with $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-action.

2. The curve Y

There are two versions of the curve.

- X^{ad} adic version analog of p-adic Reimann surface,
- X schematical version analog of a proper smooth algebraic curve.

There is an analytification morphism (GAGA) $X^{\mathrm{ad}} \to X$ and an "ample" line bundle $\mathcal{O}(1)$ on X^{ad} such that

$$X = \mathrm{Proj}(\bigoplus_{d\geq 0} \mathrm{H}^0(X^{\mathrm{ad}}, \mathcal{O}(d))).$$

Both rely on the construction of an intermediate adic space Y endowed with a "crystalline" Frobenius φ.

Let C be a complete algebraically closed field of characteristic 0. Define the tilt $C^\flat$ the inverse limit of C with respect to Frobenius, which is an algebraically closed field of characteristic p. Let B^+_{dR} be the completion of $\mathbb{A}_{\mathrm{inf}} = W(\mathcal{O}_{C^\flat})$ with repect to $(p - [p^\flat])$ with quotient field B_{dR}, A_{cris} the completion of divided power of $\mathbb{A}_{\mathrm{inf}}$ and $B_e = B_{\mathrm{cris}}^{\varphi=1}$.

The p-adic comparison theorems for crystalline/de Rham/étale cohomology lead one to consider the category of pairs (W_e, W^+_{dR}) where W_e is a free B_e-module and W^+_{dR} is a free B^+_{dR}-module such that

$$B_{\mathrm{dR}} \otimes_{B_e} W_e = B_{\mathrm{dR}} \otimes_{B^+_{\mathrm{dR}}} W^+_{\mathrm{dR}}.$$

We will construct a curve X such that $B_e = \mathcal{O}(X - \{\infty\}), B^+_{\mathrm{dR}} = \mathcal{O}_{X,\infty}$. The fundamental exact sequence

$$0 \to \mathbb{Q}_p \to B_e \to B_{\mathrm{dR}}/B^+_{\mathrm{dR}} \to 0$$

tells us the sections. The category of (W_e, W^+_{dR}) corresponds to the category of vector bundles over X. Since $B_e = B_{\mathrm{cris}}^{\varphi=1}$, this suggests

$$X^{\mathrm{ad}} = Y^{\mathrm{ad}}/\varphi^{\mathbb{Z}}$$

where $Y^{\mathrm{ad}} = \mathrm{Spa}(A_{\mathrm{inf}}) - (p[p^\flat])$.

In general, let E be a discretely valued non-archemedean field with uniformizer π with finite residue field $\mathbb{F}_q = \mathcal{O}_E/\pi$. Let $F/\mathbb{F}_q$ be a perfectoid field, i.e., a perfect field, complete with respect to a non-trivial absolute value $|\cdot| : F \to \mathbb{R}_{\geq 0}$. We will attach to this data a curve $X_{F,E}/E$. More generally, we can define "a family of curves"

$$X_S = (X_{k(s)})_{s\in|S|}$$

for perfectoid $S/\mathbb{F}_q$. If G is a reductive group over E, one can define a stack

$$\mathrm{Bun}_G : S \to \{G\text{-bundles on } X_S\}.$$

We will study the perverse ℓ-adic sheaves on Bun_G.

2.1. **Affinoid space and adic space.** Let's recall the definition of adic spaces. This is not a prt of the lectures. Let k be a nonarchimedean field and R a topological k-algebra.

Definition 2.1. (1) If there is a subring $R_0 \subset R$ such that $\{aR_0\}_{a\in k^\times}$ forms a basis of open neighborhoods of 0, it's called a *Tate k-algebra*. A subset $M \subset R$ is called *bounded* if $M \subset aR_0$ for some $a \in k^\times$.
(2) An *affinoid k-algebra* is a pair (R, R^+) consisting of a Tate k-algebra R and open integrally closed subring $R^+ \subset R^\circ$.
(3) An affinoid k-algebra (R, R^+) is said to be *tft* if R is a quotient of $k\langle T_1, \dots, T_n\rangle$ for some n and $R^+ = R^\circ$.

Definition 2.2. Denote by $X = \mathrm{Spa}(R, R^+)$ the set of equivalent classes of continuous valuations on R, which is ≤ 1 on R^+. We equip X the topology which has open *rational subsets*

$$U\left(\frac{f_1, \dots, f_n}{g}\right) = \{x \in X \mid |f_i(x)| \le |g(x)|, \forall x \in X\}$$

as basis, where $f_1, \dots, f_n$ generates R.

Definition 2.3. A topological space X is called *spectral* if it satisfies the following equivalent properties.
(1) There is some ring A such that $X \cong \mathrm{Spec}A$.
(2) X is an inverse limit of finite T_0 spaces.
(3) X is quasicompact, has a quasicompact topological basis, stable under finite intersections, and every irreducible closed subset has a unique generic point.

Theorem 2.4. *The space* $\mathrm{Spa}(R, R^+)$ *is spectral and* $\mathrm{Spa}(R, R^+) \cong \mathrm{Spa}(\widehat{R}, \widehat{R}^+)$.

Theorem 2.5. (1) *If $X = \emptyset$, then $\widehat{R} = 0$.*
(2) *If R is complete and $|f(x)| \ne 0, \forall x \in X$, then f is invertible.*
(3) *If $|f(x)| \le 1, \forall x \in X$, then $f \in R^+$.*

Consider the topological algebra $R[f_1 g^{-1}, \dots, f_n h^{-1}] \subset R[g^{-1}]$ and denote by B the integral closure of $R^+[f_1 g^{-1}, \dots, f_n g^{-1}]$ in it, then $(R[f_1 g^{-1}, \dots, f_n g^{-1}], B)$ is an affinoid k-algebra with completion $(R\langle f_1 g^{-1}, \dots, f_n g^{-1}\rangle, \widehat{B})$. Then

$$\mathrm{Spa}(R\langle f_1 g^{-1}, \dots, f_n g^{-1}\rangle, \widehat{B}) \to \mathrm{Spa}(R, R^+)$$

factors through $U\left(\frac{f_1,\dots,f_n}{g}\right)$ and it satisfies the corresponding universal property. Define presheaves

$$(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) = (R\langle f_1 g^{-1}, \dots, f_n g^{-1}\rangle, \widehat{B})$$

and on general W,

$$\mathcal{O}_X = \varprojlim_{U\subset W} \mathcal{O}_X(U).$$

Moreover $U \cong \mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$.

The stalk $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $\{f \mid f(x) = 0\}$ and $\mathcal{O}_{X,x}^+$ is a local ring with maximal ideal $\{f \mid f(x) < 1\}$.

Definition 2.6. We call R is *strongly neotherian* if $\widehat{R}\langle T_1, \dots, T_n\rangle$ is noetherian for any n.

Theorem 2.7. *If R is strongly neotherian, then $\mathcal{O}_X$ is a sheaf.*

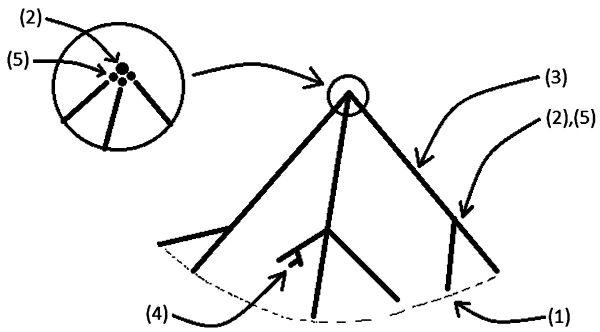
Definition 2.8. Consider triple $(X, \mathcal{O}_X, (|\cdot(x)|, x \in X))$ where $(X, \mathcal{O}_X)$ is a locally ringed space and $|\cdot(x)|$ is a continuous valuation on $\mathcal{O}_{X,x}$ for any $x \in X$. Such triple isomophic to $\mathrm{Spa}(R, R^+)$ where $\mathcal{O}_X$ is a sheaf is called an affinoid adic space.

It is called an *adic space* if it's locally an affinoid adic space.

Proposition 2.9. *For affinoid adic space $X = \mathrm{Spa}(R, R^+)$ and any adic space Y over k,*

$$\mathrm{Hom}(Y, X) = \mathrm{Hom}((\widehat{R}, \widehat{R}^+), (\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y))).$$

Example 2.10. Assume that k is complete and algebraically closed. Let $R = k\langle T\rangle$ and $R^+ = R^\circ = k^\circ\langle T\rangle$. Fix a norm $|\cdot| : k \to \mathbb{R}_{\geq 0}$. Then $X = \mathrm{Spa}(R, R^+)$ consists of



(1) The classical point. For $x \in k^\circ$,

$$R \longrightarrow \mathbb{R}_{\geq 0}$$

$$f = \sum a_n T^n \longmapsto |f(x)| = |\sum a_n x^n|.$$

(2)(3) The rays of the tree. For $0 \leq r \leq 1, x \in k^\circ$,

$$R \longrightarrow \mathbb{R}_{\geq 0}$$

$$f = \sum a_n (T-x)^n \longmapsto \sup |a_n| r^n = \sup_{y \in k^\circ, |y-x| \leq r} |f(y)|.$$

If $r = 0$, it is the classical point. If $r = 1$, it doesnot depend on x, which is called the Gausspoint.

If $r \in |k^\times|$, it's said to be of type (2), otherwise of type (3).

(4) Dead ends of the tree. Let $D_1 \supset D_2 \supset \cdots$ be a sequence of disks with $\cap D_i = \emptyset$. It occurs when k is not spherically complete.

$$R \longrightarrow \mathbb{R}_{\geq 0}$$

$$f \longmapsto \inf_i \sup_{x \in D_i} |f(x)|.$$

(5) For $\Gamma = \mathbb{R}_{\geq 0} \times \gamma^{\mathbb{Z}}$, where $\gamma = r^-$ or $r^+ (r < 1)$.

$$R \longrightarrow \Gamma \cup \{0\}$$

$$f = \sum a_n (T-x)^n \longmapsto \sup |a_n| \gamma^n.$$

This only depends on the disc $D(x, < r)$ or $D(x, r)$. Thus if $r \notin |k^\times|$, it's of type (3). Every rays of point of type (2) correspond a valuation of type (5).

2.2. **Holomorphic function of the variable** p. Let E be a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. As a comparison, we also take $E = \mathbb{F}_q[[t]]$. It is the coefficient field of the p-adic Hodge theory.

Definition 2.11. Define

$$\mathbb{A} = \mathbb{A}_{\mathrm{inf}} = \begin{cases} W_{\mathcal{O}_E}(\mathcal{O}_F) = W(\mathcal{O}_F) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E, & E/\mathbb{Q}_p, \\ \mathcal{O}_F \widehat{\otimes}_{\mathbb{F}_q} \mathcal{O}_E = \mathcal{O}_F[[\pi]], & E = \mathbb{F}_q[[t]]. \end{cases}$$

Then

$$\mathbb{A} = \left\{ \sum_{n \geq 0} [x_n] \pi^n \,\middle|\, x_n \in \mathcal{O}_F \right\}.$$

Fix $\varpi \in F$ with $0 < |\varpi| < 1$. Then $\mathbb{A}$ is complete under the $(\pi, [\varpi])$-adic topology. Consider the adic space $\mathrm{Spa}(\mathbb{A}, \mathbb{A})$. It has only one closed point with kernel $(\pi, \mathfrak{m}_F)$. Define

$$\mathcal{Y} = \mathrm{Spa}(\mathbb{A}, \mathbb{A})_a = \mathrm{Spa}(\mathbb{A}, \mathbb{A}) \backslash \{\text{closed point}\} = \mathrm{Spa}(\mathbb{A}, \mathbb{A}) \backslash V(\pi, [\varpi])$$

and an open subspace

$$Y = \mathrm{Spa}(\mathbb{A}, \mathbb{A}) \backslash V(\pi[\varpi]).$$

Here the subscript a indicates we take the analytic points and $\mathcal{Y}$ is not affinoid.

Consider the space of holomorphic functions $\mathcal{O}(Y)$. Let

$$\mathbb{A}\left[\frac{1}{\pi}, \frac{1}{[\varpi]}\right] = \left\{ \sum_{n \gg -\infty} [x_n]\pi^n \,\middle|\, x_n \in F, \sup |x_n| < +\infty \right\}$$

be the set of holomorphic functions on Y that are meromorphic along $(\pi), ([\varpi])$. For $\rho \in (0,1), f = \sum_{n \gg -\infty} [x_n]\pi^n$, define the Gauss norms

$$|f|_\rho := \sup_n |x_n|\rho^n = \sup_{|y| \le \rho} f(y).$$

Proposition 2.12. *The space*

$$B = \mathcal{O}(Y)$$

is the completion of $\mathbb{A}\left[\frac{1}{\pi}, \frac{1}{[\varpi]}\right]$ *with respect to* $\{|\cdot|_\rho\}$.

For compact subset $I \subset (0,1)$, the completion B_I with respect to $\{|\cdot|_{\rho \in I}\}$ is a Banach E-algebra and

$$B = \varprojlim_{I \subset (0,1)} B_I$$

is a Fréchet space. In particular, if $I = [\rho_1, \rho_2]$, B_I is the completion with respect to $\{|\cdot|_{\rho_1}, |\cdot|_{\rho_2}\}$.

In the case $E = \mathbb{F}_q[[\pi]]$,

$$Y = \mathbb{D}_F^* = \{0 < |\pi| < 1\} \subset \mathbb{A}_F^1$$

and

$$B = \mathcal{O}(Y) = \left\{ \sum_{n \gg -\infty} x_n \pi^n \,\middle|\, x_n \in F, \lim_{n \to +\infty} |x_n|\rho^n = 0, \forall \rho \right\}.$$

We have natural maps

$$\begin{tikzcd} & \mathbb{D}_F^* \arrow[dl] \arrow[dr] & \\ \mathrm{Spa}(F) & & \mathbb{D}_{\mathbb{F}_q}^* = \mathrm{Spa}(\mathbb{F}_q((\pi))) = \mathrm{Spa}(E). \end{tikzcd}$$

The map on the left is locally of finite type, but $\mathbb{D}_F^* \to \mathrm{Spa}(E)$ is not.

Remark 2.13. If $(x_n) \in F^{\mathbb{Z}}$ such that $\lim_{|n| \to +\infty} |x_n|\rho^n = 0, \forall \rho$, then $\sum [x_n]\pi^n \in B$. But not every element can be written in this form.

2.3. **Newton polygon.**

Proposition 2.14. $|fg|_\rho = |f|_\rho |g|_\rho$, *i.e.*, $|\cdot|$ *is a valuation.*

For $\rho = q^{-r}, r \in (0, +\infty), |f|_\rho = q^{-v_r(f)}$, where

$$v_r(f) := \inf(v(x_n) + nr).$$

Here $v = -\log_q |\cdot|$ on F. Then $r \mapsto v_r(f)$ is a convex function.

In the case $E=\mathbb{F}_q[[\pi]]$, $f=\sum x_n\pi^n\in\mathcal{O}(Y)$ defines a Newton polygon $\mathrm{Newt}(f)$ the decreasing convex hull of $\{(n,v(x_n))\}$. Then positive slopes of $\mathrm{Newt}(f)$ one-to-one correspond to the set of valuations of roots of F on $\mathbb{D}_F^*$.

Assume $E/\mathbb{Q}_p$. Recall the Legendre transform gives a bijection between the set of convex decreasing function $\mathbb{R}\to\mathbb{R}\cup\{\infty\}$, $\not\equiv+\infty$ and the set of concave function $(0,+\infty)\to\mathbb{R}\cup\{-\infty\},\not\equiv-\infty$ as

$$\mathcal{L}(\varphi)(r)=\inf\nolimits_{t\in\mathbb{R}}(\varphi(t)+tr),$$
$$\mathcal{L}^{-1}(\psi)(t)=\sup_{r\in(0,\infty)}(\psi(r)-tr).$$

Proposition 2.15. *For convex decreasing function $f,g:\mathbb{R}\to\mathbb{R}\cup\{\infty\}$, we have*

$$\mathcal{L}(f\circledast g)=\mathcal{L}(f)+\mathcal{L}(g),$$

where

$$(f\circledast g)(x)=\inf\nolimits_{a+b=x}(f(a)+g(b)).$$

The Legendre transform maps polygons to polygons, and the slopes of φ (resp. ψ) one-to-one correspond to the x-coordinates of break points of $\mathcal{L}(\varphi)$ (resp. $\mathcal{L}^{-1}(\psi)$).

Proposition 2.16. *For nonzero $f\in B$, there is a sequence $\{f_n\}$ in $\mathbb{A}\left[\frac{1}{\pi},\frac{1}{[\varpi]}\right]$ tending to f. Then for any compact subset $K\subset(0,+\infty)$, there is an integer N such that for any $n\geq N$, $v_r(f)=v_r(f_n)$ for any $r\in K$.*

As a corollary, the convex function $r\mapsto v_r(f)$ is a polygon with integral slopes.

Define

$$\mathrm{Newt}(f):=\mathcal{L}^{-1}(r\mapsto v_r(f)).$$

Then

$$\mathrm{Newt}(fg)=\mathrm{Newt}(f)\circledast\mathrm{Newt}(g).$$

Let $I\subset(0,1)$ be a compact subset and $0\neq f\in B_I$. Denote by $\mathrm{Newt}_I(f)$ the part of Newton polygon consisting of the slope in $-\log_q(I)$ part. But $\{v_r(f)\}_{r\in-\log_q(I)}$ do not determine $\mathrm{Newt}_I(f)$. For example, $I=\{q^{-r}\}$, we need to know the left and right break point of the slope r part to determine $\mathrm{Newt}_I(f)$.

Denote by ∂_l,∂_r the left/right derivation. Then $(v_r(f),\partial_l v_r(f),\partial_r v_r(f))_{r\in-\log_q(I)}$ determine $\mathrm{Newt}_I(f)$. The rank 2 valuations with image in $\mathbb{R}\times\mathbb{Z}$

$$f\mapsto(v_r(f),-\partial_l v_r(f)),$$

$$f\mapsto(v_r(f),\partial_r v_r(f)),$$

are specializations of v_r.

2.4. **Zeros of holomorphic functions.** Recall Jensen's inequality/equality. For nonzero $f\in\mathcal{O}(\mathbb{C})$ such that $f(0)\neq0$. Let $R>0$ such that f has no zero on $\{|z|=R\}$. Let $a_1,\dots,a_n$ be zeros of f in $\{|z|<R\}$. Then

$$\ln|f(0)|=\frac{1}{2\pi}\int_0^{2\pi}\ln|f(Re^{i\theta})|\mathrm{d}\theta-n\ln R+\sum_{i=1}^n\ln|a_i|$$

and

$$\ln|f(0)|\leq M(R)-n\ln R+\sum_{i=1}^n\ln|a_i|,$$

where $M(R)$ is the maximal modulus on $\{|z|=R\}$.

In the non-zrchimedead setting, there is an equality. Assume $E = \mathbb{F}_q[[\pi]]$. For nonzero $f = \sum_{n\geq 0} x_n \pi^n \in \mathcal{O}(\mathbb{D}_F)$, $f(0) = x_0 \neq 0$. Assume it has roots $(a_i)_{i\geq 1}$ with $v(a_1) \geq v(a_2) \geq \dots$. Then the slopes of $\mathrm{Newt}(f)$ are valuations of roots of f,

$$v(f(0)) = v_r(f) - nr + \sum_{i=1}^{n} v(a_i).$$

We want to do the same for $E = \mathbb{Q}_p$. We need to define zeros of f in this setting.

For $E = \mathbb{F}_q((\pi))$,

$$Y = \mathbb{D}_F^* = \{0 < |\pi| < 1\}$$

and

$$\begin{aligned}
|Y|^{\mathrm{cl}} &= \left\{ z \in \bar{F} \mid 0 < |z| < 1 \right\} / \mathrm{Gal}(\bar{F}/F) \\
&= \{ P \in F[\pi] \mid \text{irreducible with all roots such that } 0 < |z| < 1 \} / F^\times \\
&= \{ P \in \mathcal{O}_F[\pi] \mid \text{unitary irreducible such that } 0 < |P(0)| < 1 \}.
\end{aligned}$$

Definition 2.17. $f = \sum_{n\geq 0} x_n \pi^n \in \mathbb{A}$ is (distinguished) primitive of degree $d > 0$ if $x_0 \neq 0, x_0, \dots, x_{d-1} \in \mathfrak{m}_F, x_d \in \mathcal{O}_F^\times$.

By Weierstrass fatorization, $f = uP$ uniquely where $u \in \mathcal{O}_F[[\pi]]^\times$ and $P \in \mathcal{O}_F[\pi]$ is unitary with degree d. Thus

$$|Y|^{\mathrm{cl}} = \{\text{primitive irreducible elements}\} / \mathcal{O}_F[[\pi]]^\times.$$

Assume $E/\mathbb{Q}_p$.

Definition 2.18. $f = \sum_{n\geq 0} [x_n] \pi^n \in \mathbb{A}$ is primitive of degree d if $x_0 \neq 0, x_0, \dots, x_{d-1} \in \mathfrak{m}_F, x_d \in \mathcal{O}_F^\times$.

It's equivalent to say, $f \bmod \pi \neq 0$ in $\mathcal{O}_F$ and $f \bmod W_{\mathcal{O}_E}(\mathcal{O}_F) \neq 0$ in $W_{\mathcal{O}_E}(k_F)^d$. The degree of f is $v_\pi(f \bmod W_{\mathcal{O}_E}(\mathcal{O}_F))$. Thus $\deg(fg) = \deg f + \deg g$.

Definition 2.19.

$$|Y|^{\mathrm{cl}} = \{\text{irreducible primitive}\} / \mathbb{A}^\times.$$

We will show that this is the set of the classical points of Y.

2.5. **Perfectoid fields and tilting.**

Definition 2.20. A complete field K with respect to a norm $|\cdot| : K \to \mathbb{R}_{\geq 0}$ is called a perfectoid field, if there is an element $\varpi \in K$ such that $|p| \leq |\varpi| < 1$ such that $\mathrm{Frob} : \mathcal{O}_K/\varpi \to \mathcal{O}_K/\varpi$ is surjective.

For example, $\widehat{\mathbb{Q}(\zeta_{p^\infty})}(p > 2), \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$. An algebraic closed complete valued field is perfectoid. In char p case, K is perfectoid if and only if it is perfect.

Let K be a perfectoid field. Define the *tilting*

$$K^\flat = \varprojlim_{x \mapsto x^p} K = \left\{ (x^{(n)})_{n\geq 0} \in K^{\mathbb{N}} \mid (x^{(n+1)})^p = x^{(n)} \right\},$$

with

$$(xy)^{(n)} = x^{(n)} y^{(n)}, \quad (x+y)^{(n)} = \lim_{k\to +\infty} (x^{(n+k)} + y^{(n+k)})^{p^k}.$$

Define

$$x^\# := x^{(0)}$$

and

$$\begin{aligned}
|\cdot| : K^\flat &\longrightarrow \mathbb{R}_{\geq 0} \\
x &\longmapsto |x^\#|.
\end{aligned}$$

Then $K^\flat$ is also perfectoid. Moreover, there is an isomorphism

$$\begin{aligned} \mathcal{O}_{K^\flat} &\xrightarrow{\sim} \varprojlim_{x\mapsto x^p} \mathcal{O}_K/p \\ x &\mapsto (x^{(n)} \bmod p)_{n\geq 0} \\ \lim_{k\to+\infty} (\hat{y}_{n+k})^{p^k} &\leftarrow\!\shortmid (y_n)_{n\geq 0}. \end{aligned}$$

Example 2.21. If K is of characteristic p, then $K^\flat = K$.

Example 2.22. If $K = \widehat{\mathbb{Q}_p(\zeta_{p^\infty})}, \epsilon = (\zeta_{p^n})_{n\geq 0} \in K^\flat$ and $\pi_\epsilon = \epsilon - 1 \in K^\flat$, then $K^\flat = \mathbb{F}_p((\pi_\epsilon^{1/p^\infty}))$. In fact, $\mathbb{Z}_p(\zeta_{p^\infty})/p \xrightarrow{\sim} \mathbb{F}_p(\pi_\epsilon^{1/p^\infty})/\pi_\epsilon$.

If $K = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}, \pi = (p^{1/p^n})_{n\geq 0} \in K^\flat$, then $K^\flat = \mathbb{F}_p((\pi^{1/p^\infty}))$. In fact, $\mathbb{Z}_p(p^{1/p^\infty})/p \xrightarrow{\sim} \mathbb{F}_p(\pi^{1/p^\infty})/\pi$.

Remark 2.23. In fact, Fontaine gave the isomorphism

$$R^\flat = \varprojlim_{x\mapsto x^p} R/pR \xrightarrow{\sim} \left\{(x^{(n)})_{n\geq 0} \in R^{\mathbb{N}} \mid (x^{(n+1)})^p = x^{(n)}\right\}$$

for any separated complete p-adic ring R.

Theorem 2.24. *Let K be a perfectoid field. Then*

(1) *If L/K is finite, then L is perfectoid and $[L^\flat : K^\flat] = [L : K]$.*
(2) *$\mathcal{O}_L/\mathcal{O}_K$ is almost étale, i.e., if $n = [L : K], \forall 0 < \epsilon < 1, \exists e_1, \ldots, e_n \in \mathcal{O}_L$ such that*

$$\epsilon \leq |\mathrm{disc}(\mathrm{Tr}_{L/K}(e_i e_j))_{1\leq i,j,\leq n}| \leq 1.$$

(3) *$(\cdot)^\flat$ induces an equivalence between the set of finite étale K-algebras and the set of finite étale $K^\flat$-algebras.*

Corollary 2.25. (1) *K is algebraically closed if and only if $K^\flat$ is.*
(2) *$\mathrm{Gal}(\overline{K}/K) \xrightarrow{\sim} \mathrm{Gal}(\overline{K^\flat}/K^\flat)$, where $\overline{K^\flat}$ is the union of all $L^\flat$ where L/K is finite.*

Proposition 2.26. *The functors*

$$\{p\text{-adic rings}\} \underset{W(\cdot)}{\overset{(\cdot)^\flat}{\rightleftarrows}} \{\text{perfect } \mathbb{F}_p\text{-algebras}\}$$

are adjoint, i.e.,

$$\mathrm{Hom}(W(A), B) = \mathrm{Hom}(A, B^\flat).$$

The adjuncation morphisms are

$$\begin{aligned} R &\xrightarrow{\sim} W(R^\flat) \\ x &\mapsto [x^{1/p^n}], \end{aligned}$$

$$\begin{aligned} \theta : W(R^\flat) &\xrightarrow{\sim} R \\ \sum [x_n] p^n &\mapsto \sum x_n^\# p^n. \end{aligned}$$

Remark 2.27. If R is a p-adic ring such that the Frobenius on R/pR is surjective, then $\theta \bmod p$ is $R^\flat \to R/pR$. Thus θ is surjective by Nakayama lemma and R is a quotient of $W(R^\flat)$.

2.6. **Classical points.**

Theorem 2.28. *Let ξ be an irreducible primitive element of degree d and $\theta : \mathbb{A} \twoheadrightarrow \mathbb{A}/\xi = \mathcal{O}_K, K = \mathcal{O}_K[1/p]$.*

(1) *K/E is a perfectoid field with $|\theta([x])| = |x|$.*
(2) *The morphism*
$$\begin{aligned} \mathcal{O}_F &\longrightarrow \mathcal{O}_K^\flat \\ x &\longmapsto \theta([x^{p^{-n}}])_{n\geq 0} \end{aligned}$$
induces $K^\flat/F$ of degree d. In particular, $K^\flat = F$ if $d = 1$.
(3) *For $d = 1$, this induces*
$$\begin{aligned} |Y|^{\mathrm{cl},\deg=1} = \mathrm{Prim}^{\deg=1}/\mathbb{A}^\times &\xrightarrow{\sim} \left\{ K/E \textit{ perfectoid }, K^\flat = F \right\} / \sim \\ (\xi) &\mapsto (\mathbb{A}/\xi)[1/p] \\ \ker\theta &\leftarrow K/E. \end{aligned}$$

Thus any ξ defines a valuation
$$\mathbb{A}\left[\frac{1}{\pi}, \frac{1}{[\varpi]}\right] \to \mathbb{A}\left[\frac{1}{\pi}, \frac{1}{[\varpi]}\right]/\xi \xrightarrow{|\cdot|} \mathbb{R}_{\geq 0},$$
and
$$|Y|^{\mathrm{cl}} = \{V(\xi) \mid \xi \in \mathbb{A} \text{ irreducible primitive } \} \subset |Y|.$$
We see that for $y \in |Y|^{\mathrm{cl}}$, $k(y)/E$ is perfectoid and $[k(y)^\flat : F] < +\infty$.

Theorem 2.29. *Assume that F is algebraically closed.*

(1) *$\forall y \in |Y|^{\mathrm{cl}}, k(y)$ is algebraically closed.*
(2) *$\forall \xi, \deg(\xi) = 1$.*
(3) *any primitive element ξ can be written as*
$$\xi = u(\pi - [a_1]) \cdots (\pi - [a_d])$$
where $u \in \mathbb{A}^\times$.

For $y = V(\xi) \in |Y|^{\mathrm{cl}}, \xi = \sum [x_n]\pi^n$ is primitive of degree d, set
$$|\xi| = |x_0|^{1/d} = |\pi(y)|.$$
This defines the radius
$$|\cdot| : |Y|^{\mathrm{cl}} \to (0, 1).$$

Definition 2.30. For $y = V(\xi) \in |Y|^{\mathrm{cl}}$,
$$B^+_{\mathrm{dR},y} = \xi\text{-adic completion of } \mathbb{A}\left[\frac{1}{\pi}, \frac{1}{[\varpi]}\right] = \widehat{\mathcal{O}_{Y,y}}.$$
It is a discrete valuation ring with uniformizer ξ and residue field $k(y)$.

2.7. **Localization of zeros.**

Theorem 2.31. *For nonzero $f \in B$,*
$$\left\{ -\log_q |y| \mid y \in |Y|^{\mathrm{cl}}, f(y) = 0 \right\}$$
coincides the slopes of $\mathrm{Newt}(f)$.

Definition 2.32. For any interval $I \subset (0, 1)$,
$$|Y_I|^{\mathrm{cl}} = \left\{ y \in |Y|^{\mathrm{cl}} \mid |y| \in I \right\}.$$

Theorem 2.33. *For any compact subset $I \subset (0, 1)$, B_I is a PID with* $\mathrm{Spm}B_I = |Y_I|^{\mathrm{cl}}$. *In fact,* $\mathrm{Spm}B = \{(\xi) \mid |\xi| \in I\}$.

Proposition 2.34.
$$B_I^\times = \{f \in B_I \setminus \{0\} \mid \mathrm{Newt}(f) = \emptyset\}.$$

Define the *Robba ring* the local ring of $\mathcal{Y}$ at origin,
$$\mathcal{R} = \varinjlim_{\rho \to 0^+} B_{(0,\rho]}.$$
This is a Bezout ring.

Define
$$\mathrm{Div}^+(Y_I) = \{D = \sum_{y \in |Y_I|^{\mathrm{cl}}} m_y[y] \mid \mathrm{supp}(D) \text{ is locally finite }, m_y \in \mathbb{N}\},$$
and
$$\begin{aligned} \mathrm{div} : (B_I \setminus \{0\})/B_I^\times &\longrightarrow \mathrm{Div}^+(Y_I) \\ f &\longmapsto \sum \mathrm{ord}_y(f)[y]. \end{aligned}$$

Remark 2.35. If $E = \mathbb{F}_q((\pi)), I = (0,1)$, the div map is a bijection if and only if F is spherically complete (Larzard).

For any $\rho \in (0,1)$,
$$\mathrm{div} : B_{(0,\rho]} \setminus \{0\} / B_{(0,\rho]}^\times \xrightarrow{\sim} \mathrm{Div}^+(Y_{(0,\rho]}).$$
In fact, for $D = \sum_{n \geq 0} [y_n]$ with $|y_n| \to 0$, write $y_n = V(\xi_n)$, the series
$$f = \prod_{n \geq 0} \xi_n \pi^{-\deg \xi_n}$$
converges, where $\xi_n \equiv \pi^{\deg \xi} \bmod W_{\mathcal{O}_E}(\mathcal{O}_F)$.

2.8. **Parametrization of classical points.** Assume F is algebraically closed. If $E = \mathbb{F}_q((\pi))$, then $|Y|^{\mathrm{cl}} = |D_F^*|^{\mathrm{cl}} = \mathfrak{m}_F \setminus \{0\}$. Thus
$$\begin{aligned} D^*(F) = \mathfrak{m}_F \setminus \{0\} &\xrightarrow{\sim} |Y|^{\mathrm{cl}} \\ a &\longmapsto V(\pi - a). \end{aligned}$$

If $E/\mathbb{Q}_p$, $a \in \mathfrak{m}_F \setminus \{0\}, y = V(\pi - [a])$,
$$\begin{aligned} D^*(F) = \mathfrak{m}_F \setminus \{0\} &\twoheadrightarrow |Y|^{\mathrm{cl}} \\ a &\longmapsto V(\pi - a). \end{aligned}$$
But it's hard to describe fibers.

For $y \in |Y|^{\mathrm{cl}}, C_y = k(y)/E$ is algebraically closed. Choose $\underline{\pi} \in C_y^\flat$ such that $\underline{\pi}^\sharp = \pi$. Then $y = V(\pi - [\underline{\pi}])$.

Consider the case $E = \mathbb{Q}_p$. It's same for general E by using Lubin-Tate groups. Then
$$\widehat{\mathbb{G}}_m(\mathcal{O}_F) = (1 + \mathfrak{m}_F, \times)$$
is a Banach space as
$$\begin{aligned} a.\epsilon &= \sum_{k \geq 0} \binom{a}{k} (\epsilon - 1)^k, \\ p.\epsilon &= \epsilon^p \end{aligned}$$
and the fact that F is perfect.

Definition 2.36. For any $1 \neq \epsilon \in 1 + \mathfrak{m}_F$,
$$u_\epsilon := \frac{[\epsilon] - 1}{[\epsilon^{1/p}] - 1} = 1 + [\epsilon^{1/p}] + \cdots + [\epsilon^{\frac{p-1}{p}}] \in \mathbb{A}.$$

Lemma 2.37. *u_ϵ is primitive of degree* 1.

Indeed,

$$u_\epsilon \bmod p = 1 + \epsilon^{1/p} + \cdots + \epsilon^{(p-1)/p} = \frac{\epsilon - 1}{\epsilon^{1/p} - 1} \in \mathcal{O}_F$$

is nonzero,

$$u_\epsilon \bmod W(\mathfrak{m}_F) \equiv 1 + [1] + \cdots + [1] = p \in W(k_F).$$

Set

$$C_\epsilon = B/u_\epsilon = k(y)$$

where $y = V(\epsilon)$. Then $\epsilon = (\epsilon^{(n)}) \in F = C_\epsilon^\flat$, where $\epsilon^{(n)} = \theta_\epsilon([\epsilon^{1/p^n}])$. Then

$$1 + \epsilon^{(1)} + \cdots + (\epsilon^{(1)})^{p-1} = \theta_\epsilon(1 + [\epsilon^{1/p}] + \cdots + [\epsilon^{(p-1)/p}]) = \theta_\epsilon(u_\epsilon) = 0,$$

thus $\epsilon^{(1)} \in \mu_p(C_\epsilon)$. Moreover

$$\mathcal{O}_{C_\epsilon}/p\mathcal{O}_{C_\epsilon} = \mathbb{A}/(p, u_\epsilon) = \mathcal{O}_F/\bar{u}_\epsilon,$$

where

$$\bar{u}_\epsilon = \frac{\epsilon - 1}{\epsilon^{1/p} - 1} = (\epsilon - 1)^{\frac{p-1}{p}}.$$

Since $\epsilon^{(1)} - 1 \equiv \epsilon^{1/p} - 1 \bmod p$, $\epsilon^{1/p} - 1 \notin \mathcal{O}_F u_\epsilon$, $\epsilon^{(1)} - 1 \neq 0 \bmod p$ in C_ϵ. Hence $\epsilon^{(1)} \in \mu_p(C_\epsilon)$ is primitive and $\underline{\epsilon}$ is a generator of $\mathbb{Z}_p(1)(C_\epsilon) = \left\{x \in C_\epsilon^\flat \mid x^\sharp = 1\right\}$.

Proposition 2.38.

$$\begin{aligned} ((1 + \mathfrak{m}_F)\setminus\{1\})/\mathbb{Z}_p^\times &\xrightarrow{\sim} |Y|^{\mathrm{cl}} \\ \epsilon &\longmapsto V(u_\epsilon). \end{aligned}$$

The inverse is given by $y \in |Y|^{\mathrm{cl}}, C_y = k(y)/E$. Choose ϵ a basis of $\mathbb{Z}_p(1)(C_y) \hookrightarrow (C_y^\flat)^\times = F^\times$. Then $\epsilon \in (1 + \mathfrak{m}_F)\setminus\{1\}, y = V(u_\epsilon)$.

Remark 2.39. Let

$$Y^\diamond = \mathrm{Spa} F \times_{\mathrm{Spa}\mathbb{F}_p} (\mathrm{Spa}\mathbb{Q}_p)^\diamond) = \mathrm{Spa} F \times \mathrm{Spa}\mathbb{Q}_p^{\mathrm{cyc},\flat}/\mathbb{Z}_p^\times = \mathbb{D}_F^{*,1/p^\infty}/\mathbb{Z}_p^\times.$$

where $\mathbb{Z}_p^\times = \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{cyc}}/\mathbb{Q}_p)$, $\mathbb{Q}_p^{\mathrm{cyc},\flat} = \mathbb{F}_p((T^{1/p^\infty}))$. The action of $\mathbb{Z}_p^\times$ is given by $a.T = \sum_{k\geq 0}\binom{a}{k}(T-1)^k$. Then $|Y| = |Y^\diamond| = |\mathbb{D}_F^*|/\mathbb{Z}_p^\times$.

$$|Y_{\hat{\bar{F}}}|^{\mathrm{cl},G_F\text{-finite}} \twoheadrightarrow |Y_F|^{\mathrm{cl}}$$

and $|Y_F|^{\mathrm{cl}} = |Y_{\hat{\bar{F}}}|^{\mathrm{cl},G_F\text{-finite}}/G_F = ((1 + \mathfrak{m}_{\hat{\bar{F}}})\setminus\{1\})/\mathbb{Z}_p^\times)^{G_F\text{-finite}}/G_F$.

3. The curve X

The curve Y is Stein, it's completely determined by the E-Frechét algebra $\mathcal{O}(Y)$. It's *preperfectoid*, i.e., $Y\widehat{\otimes}_E K$ is perfectoid for a perfectoid field K/E. The Frobenius φ acts on $\mathbb{A}$ by

$$\sum [x_n]\pi^n \mapsto \sum [x_n^q]\pi^n.$$

This induces the action of φ on $\mathcal{O}(Y)$ and Y with $|\varphi(y)| = |y|^{1/q}$.

Theorem 3.1. *(1)* $Y\langle \frac{\pi^a}{[\varpi]^b}, \frac{[\varpi]^c}{\pi^d}\rangle = \mathrm{Spa}(R, R^\circ)$ *and* R *is an* E*-Banach algebra and a PID.*

(2) R *is strongly neotherian.*

Thus Y is a one-dimensonal regular adic space over E. Define

$$X^{\mathrm{ad}} := Y/\varphi^{\mathbb{Z}},$$

this is a quasi-compact adic space over E, neotherian regular of dimension one. For $0 < \rho_1 < \rho_2 < \rho_1^{1/q} < 1$,

$$X^{\mathrm{ad}} = Y_{[\rho_1,\rho_2]} \cup Y_{[\rho_2,\rho_1^{1/q}]}.$$

Remark 3.2. φ is the arithmetic Frobenius. For $X/\mathbb{F}_q$, there are geometric Frobenius $\mathrm{Frob}_X \times \mathrm{Id}$, arithmetic Frobenius $\mathrm{Id} \times \mathrm{Frob}_q$ and absolute Frobenius $\mathrm{Frob}_X \times \mathrm{Frob}_q$ on $X_{\bar{\mathbb{F}}_q}$.

The line bundles on X^{ad} are $\varphi^{\mathbb{Z}}$-equivariant line bundles over Y, i.e., projective φ-modules over B of rank 1, or free $\mathcal{R}$-modules of rank 1. Thus $\mathrm{Pic}(X^{\mathrm{ad}}) = \mathbb{Z}$, where n corresponds $(B \cdot e, \varphi)$ with $\varphi(e) = \pi^{-n} e$.

Definition 3.3. Define $\mathcal{O}(d)$ corresponds $(B, \pi^{-d}\varphi)$.

For a proper smooth algebraic curve X over $\mathbb{C}$, the analytic part X^{an} is a compact Riemann surface. Conversely, given a compact Riemann surface Z, there is an ample line bundle $\mathcal{L}$ over Z, e.g., $\mathcal{O}(z)$ for $z \in Z$, Then

$$\mathrm{Proj}\big(\bigoplus_{d\geq 0} \mathrm{H}^0(Z, \mathcal{L}^{\otimes d})\big)$$

is a proper smooth algebraic curve.

We claim that $\mathcal{O}(1)$ is ample. Denote by

$$P_d = \mathrm{H}^0(X^{\mathrm{ad}}, \mathcal{O}(d)) = B^{\varphi = \pi^d}$$

and

$$P = \bigoplus_{d\geq 0} P_d.$$

We take

$$X = \mathrm{Proj} P.$$

Theorem 3.4. *(1) X is a Dedekind scheme.*

(2) There is a natural morphism of ringed spaces $X^{\mathrm{ad}} \to X$ inducing $|X^{\mathrm{ad}}|^{\mathrm{cl}} := |Y|^{\mathrm{cl}}/\varphi^{\mathbb{Z}} \xrightarrow{\sim} |X| = \{$closed points$\}$ such that

$$\widehat{\mathcal{O}}_{X,x} \xrightarrow{\sim} \widehat{\mathcal{O}}_{Y,y} = B^+_{\mathrm{dR}}(k(y))$$

if $y \mapsto x$. In particular, for any $x \in |X|$, $k(x)/E$ is perfectoid.

(3) X is complete, i.e., for any $x \in |X|$, $\deg(x) := [k(x)^\flat : F]$, then $\deg(\mathrm{div}(f)) = 0$ for any $f \in E(X)^\times$. This implies we may define degree of vector bundles.

(4) There is an isomorphism

$$\begin{aligned} |X|^{\deg = 1} &\longrightarrow \{\textit{untilts of } F\}/\mathrm{Frob}^{\mathbb{Z}} \\ x &\longmapsto k(x). \end{aligned}$$

(5) If F is ac, $\infty \in |X|$, there is $t \in \mathrm{H}^0(X, \mathcal{O}(1)) \setminus \{0\}$ such that $V(t) = \{\infty\}$ and $X \setminus \{\infty\} = \mathrm{Spec} B_e$, where $B_e := B[1/t]^{\varphi = 1}$.

B_e is a PID and $(B_e, -\mathrm{ord}_\infty)$ is non-Euclidean but almost Euclidean, i.e., for any x, y, there is $x = ay + b$ with $\deg(b) \leq \deg(y)$. That's because $\mathrm{H}^1(X, \mathcal{O}_X(-1)) \neq 0$ but $\mathrm{H}^1(X, \mathcal{O}_X) = 0$.

We are going to prove that X is a curve. We assume that F is algebraically closed. The case of general perfectoid F is treated by Galois descent from $\widehat{F}$ to F.

3.1. **The fundamanetal exact sequence.**

Proposition 3.5. *P is a graded fractional ring with irreducible elements of degree 1.*

For any $0 \neq t \in P_1$, $P[1/t]_0$ is fractional with irreducible elemenets $\{t'/t \mid t' \in P_1 - Et\}$.

Proposition 3.6. *Let $t_1, \dots, t_d \in P_1 \setminus \{0\}$ associate $y_1, \dots, y_d \in |Y|^{\mathrm{cl}}$, i.e., $\mathrm{div}(t_i) = \sum_{n\in\mathbb{Z}}[\varphi^n(y_i)]$ on Y. Let $y_i = V(a_i)$, where a_i is primitive of degree 1. Then the sequence*

$$0 \to E \cdot \prod_{i=1}^{d} t_i \to B^{\varphi=\pi^d} \to B/a_1 \cdots a_d B \to 0$$

is exact.

For example, if $t \in P_1 \setminus \{0\}$,

$$0 \to E \cdot t^d \to B^{\varphi=\pi^d} \to B^+_{\mathrm{dR},y}/\mathrm{Fil}^d B^+_{\mathrm{dR},y} \to 0$$

for $y \in |Y|^{\mathrm{cl}}, t(y) = 0$.

Proof. Exactness in the middle. Suppose $f \in B^{\varphi=\pi^d} \cap a_1 \dots a_d B$ is nonzero, then

$$\mathrm{div}(f) \geq \sum_{i=1}^{d}[y_i]$$

in $\mathrm{Div}^+(X^{\mathrm{ad}})$. Then

$$\mathrm{div}(f) \geq \sum_{n\in\mathbb{Z}}\sum_{i=1}^{d}[\varphi^n(y_i)] = \mathrm{div}(\prod_{i=1}^{d} t_i)$$

and then $f = x \prod_{i=1}^{d} t_i$ for some $x \in B^{\varphi=1} = E$.

Surjectivity. We only need to prove $d = 1$ case. For any $x \in C$, $p^n x$ lies in the convergence domain of exp for $n \gg 0$. Since C is algebraically closed, there is $z \in C$ such that $\exp(p^n x) = z^{p^n}$, thus $\log z = x$ and $\log : 1 + \mathfrak{m}_C \to C$ is surjective.

Assume $E = \mathbb{Q}_p$. Let a be a primitive element of degree 1, $y = V(a)$ and $C = C_y = B/aB$. Then $C^\flat = F$. For any $\varepsilon \in 1 + \mathfrak{m}_F$, $\log([\varepsilon]) \in B^{\varphi=p}$ and $\theta(\log([\varepsilon])) = \log(\theta([\varepsilon])) = \log \varepsilon^\sharp$, $\varepsilon^\sharp \in 1 + \mathfrak{m}_C$. Take ε such that $\varepsilon^\sharp = z$, then $\theta(\log([\varepsilon])) = x$. It's same for general E by using $\log_{\mathcal{LT}}$ for Lubin-Tate group with respect to (q, π). $\square$

We are going to use the fundamental exact sequence to prove that X is a curve. Reciprocally, once the curve is constructed, we can find back the fundamental exact sequence by applying $\mathrm{H}^0(X, -)$ on

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\times \prod_{i=1}^{d} t_i} \mathcal{O}_X(d) \longrightarrow \mathcal{F} \longrightarrow 0$$

and $\mathrm{H}^1(X, \mathcal{O}_X) = 0$.

Corollary 3.7. *For any $t \in P_1 \setminus \{0\}$, $t(y) = 0, y \in |Y|^{\mathrm{cl}}$, $C = C_y$,*

$$\begin{aligned} P/tP &\longrightarrow \{f \in C[T] \mid f(0) \in E\} \\ x \bmod tP_{i-1} &\longmapsto \theta_y(x)T^i \end{aligned}$$

is an isomorphism betweem graded rings, where

$$P/tP = E \oplus \bigoplus_{i\geq 1} P_i/tP_{i-1}.$$

3.2. **Vector bundles.** Let X be an integral Dedekind scheme with generic point η. Let $\infty \in |X|$ be a closed point. Let $K = \mathcal{O}_{X,\eta}$ be the field of rational functions on X. We suppose that $X - \{\infty\}$ is affine, i.e., $X - \{\infty\} = \mathrm{Spec} A$ where $A = RH^0(X\{\infty\}, \mathcal{O}_X)$. Let t be a uniformizer in $\mathcal{O}_{X,\infty}$.

Denote by Bun_X the category of vector bundles on X, i.e., locally free $\mathcal{O}_X$-modules of finite rank.

Denote by C the category of triples (M, W, u), where M is a projective A-module of finite type, W is a free $\widehat{\mathcal{O}}_{X,\infty}$-module of finite type and

$$u : M \otimes_A \widehat{\mathcal{O}}_{X,\infty}[1/t] \xrightarrow{\sim} W[1/t]$$

is an isomorphism.

Theorem 3.8. *There is an equivalence of categories*

$$\mathsf{Bun}_X \xrightarrow{\sim} \mathsf{C}$$
$$\mathcal{E} \longmapsto (\Gamma(X - \{\infty\}, \mathcal{E}), \widehat{\mathcal{E}}_\infty, \mathrm{can}).$$

Here can *is induced by* $\Gamma(X - \{\infty\}, \mathcal{E}) \hookrightarrow \mathcal{E}_\eta = \mathcal{E}_\infty[1/t]$.

Moreover, if $\mathcal{E}$ corresponds to (M, W, u), then $\Gamma(X, -)$ on $\mathcal{E}$ has a resolution

$$\Gamma(X, \mathcal{E}) \to M \oplus W \xrightarrow{\partial} W[1/t],$$

where $\partial(m, w) = u(m) - w$. Thus

$$\mathrm{H}^0(X, \mathcal{E}) = u(M) \cap W, \qquad \mathrm{H}^1(X, \mathcal{E}) = \frac{W[1/t]}{W + u(M)}.$$

Suppose X is complete. Then there is a map $\deg : |X| \to \mathbb{N}_+$ such that $\deg(\mathrm{div}(f)) = 0$. Assume $\deg \infty = 1$. Then

$$\mathrm{H}^0(X, \mathcal{O}_X) = \left\{f \in K^\times \mid \mathrm{div}(f) \geq 0\right\} \cup \{0\} = \left\{f \in K^\times \mid \mathrm{div}(f) = 0\right\} \cup \{0\}$$

is a field. Denote by $E = \mathrm{H}^0(X, \mathcal{O}_X) \subset K$.

Denote by

$$\deg = -\mathrm{ord}_\infty : A \to \mathbb{N} \cup \{\infty\}.$$

Then $E = A^{\deg \leq 0} = A^{\deg = 0}$. Note that $A^{\deg \leq d} = \mathrm{H}^0(X, \mathcal{O}_X(d[\infty]))$ and ths sheaf $\mathcal{O}_X(d[\infty])$ corresponds $(A, t^{-d}\widehat{\mathcal{O}}_{X,\infty}, \mathrm{can})$,

$$\mathrm{H}^1(X, \mathcal{O}_X(d[\infty])) = \frac{K}{t^{-d}\mathcal{O}_{X,\infty} + A}.$$

In particular,

$$\mathrm{H}^1(X, \mathcal{O}_X(-\infty)) = \frac{K}{t\mathcal{O}_{X,\infty} + A}$$

is zero iff A is *Euclidean*, i.e., for any $x, y \in A$ with $y \neq 0$, there is $a \in A$ such that $\deg(x/y - a) < 0$. $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ iff A is *almost Euclidean*, i.e., $\deg(x/y - a) \leq 0$.

Now for our X, $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ but $\mathrm{H}^1(X, \mathcal{O}_X(-1)) \neq 0$, since B_e is almost Euclidean but not Euclidean.

3.3. **Harder-Narasimhan filtrations.** See Yves André, *Slope filtrations* `https://arxiv.org/abs/0812.3921`.

Consider

- an exact category C,
- an abelian category$\mathsf{A}$,
- an exact faithful functor $\mathcal{F} : \mathsf{C} \to \mathsf{A}$, called *generic fiber functor*, such that for any $X \in \mathsf{C}$, $\mathcal{F}$ induces an equivalence between strict sub-objects of X and sub-objects of $\mathcal{F}(X)$, the inverse functor is called schematical closure.

- an additive map $\mathrm{rk} : \mathrm{Obj}(\mathsf{C}) \to \mathbb{N}$, i.e., it factors through $\mathrm{K}_0(\mathsf{C}) \to \mathbb{Z}$, such that $\mathrm{rk}(X) = 0$ iff $X = 0$,
- an additive map $\deg : \mathrm{Obj}\mathsf{C} \to \mathbb{R}$ such that for $u : X \to Y$, if $\mathcal{F}(u)$ is an isomorphism, then $\deg X \leq \deg Y$ with equality iff u is an isomorphism.

Example 3.9. Let X be a complete integral Dedekind scheme. Then $k(X) = \mathcal{O}_{X,\eta}$ is equipped with $\deg : |X| \to \mathbb{N}_{\geq 1}$ such that for any $f \in k(X)^\times, \deg(\mathrm{div}(f)) = 0$. Take $\mathsf{C} = \mathsf{Bun}_X, \mathsf{A} = \mathsf{Vect}_{k(X)}$, $\mathcal{F}(\mathcal{E}) = \mathcal{E}_\eta$. Then the strict sub-objects of $\mathcal{E}$ are locally direct factors $\mathcal{F} \subset \mathcal{E}$. For any $V \subset \mathcal{E}_\eta$, $\mathcal{E} \cap V$ is a strict sub-object of $\mathcal{E}$.

The degree map induces $\deg : \mathrm{Pic}(X) \to \mathbb{Z}$ and then $\deg : \mathsf{Bun}_X \to \mathbb{Z}$ via $\deg(\mathcal{E}) := \deg(\det \mathcal{E})$. Then if $u : \mathcal{E} \to \mathcal{E}'$ induces an isomorphism $\mathcal{E}_\eta \xrightarrow{\sim} \mathcal{E}'_\eta$, then

$$0 \to \mathcal{E} \to \mathcal{E}' \to \mathcal{F} \to 0$$

with torsion $\mathcal{F}$, and $\deg \mathcal{E}' = \deg \mathcal{E} + \deg \mathcal{F}$. $\mathcal{F}$ can be written as $\mathcal{F} = \oplus i_{x*} M_x$ where M_x is finite length $\mathcal{O}_{X,x}$-module and

$$\deg \mathcal{F} = \sum \mathrm{length}_{\mathcal{O}_x}(M_x) \deg(x).$$

Example 3.10. If $\mathsf{C} = \mathsf{A}$ is an abelian category, $\mathcal{F} = \mathrm{Id}$, we require additive maps deg and rk, such that $\mathrm{rk}(X) = 0$ iff $X = 0$.

Example 3.11. Let k be a field, $\mathsf{BT}_k \otimes \mathbb{Q}$ is the category of p-divisible groups over k up to isogeny. This is an abelian category. We take rk to be the height and deg the dimension of associated formal group. Then the Harder-Narasimhan filtration in this category is the slope filtration. For example,

$$0 \to H^\circ \to H \to H^{\text{ét}} \to 0$$

is part of this filtration.

Example 3.12. Let L/K be an extension. Let C be the category of vector spaces V over K with a finitely decreasing fitration on V_L. The exactness should be strictly compatible with fltrations. Define

$$\mathrm{rk} = \dim_K V \qquad \deg = \sum i \cdot \dim \mathrm{gr}^i \mathrm{Fil} V_L.$$

Define $\mathcal{F} : \mathsf{C} \to \mathsf{Vect}_K$ to be the forgetful functor. Then the deserved property follows from

$$\deg = N \dim V + \sum_{i<N} \dim \mathrm{Fil}^i V_L, \quad N \ll 0.$$

Example 3.13. Let k be a perfect field with characteristic p, σ the Frobenius on $K_0 = W(k)_\mathbb{Q}$. Let K/K_0 be a finite ramified extension. Denote by φ-ModFil_{K/K_0} the category of $(D, \varphi, \mathrm{Fil} D_K)$ where (D, φ) is an isocrystal. Denote by $\mathrm{rk} = \dim_{K_0} D, \deg = t_H - t_N$. Then semi-stable slope 0 objects are weakly admissible filtered isocrystals.

Example 3.14. Let $\mathcal{R}$ be a Bezout ring, $\mathcal{E} \subset \mathcal{R}$ a field with a nontrivial valuation $v : \mathcal{E} \to \mathbb{R} \cup \{-\infty\}$. Let σ be an endomorphism that stabilizes $\mathcal{E}$ such that $v(\sigma(x)) = v(x)$. We assume that $\mathcal{E}^\times = \mathcal{R}^\times$ and for any nonzero $x \in \mathcal{R}$ such that $x^{\sigma-1} \in \mathcal{E}^\times$, $v(x^{\sigma-1}) \geq 0$. Denote by C the category of (M, φ), where M is a free $\mathcal{R}$-module with finite rank, φ is a σ-semilinear endomorphism on M such that $\varphi \otimes \mathrm{Id} : M^{(\sigma)} \xrightarrow{\sim} M$. Denote $\mathcal{F}(M, \varphi) = (M \otimes_\mathcal{R} \mathrm{Frac}\mathcal{R}, \varphi \otimes \sigma)$, $\mathrm{rk} = \mathrm{rk}_\mathcal{R}(M), \deg = -v(\det \varphi) = -v(a)$, where $\det(M, \varphi) = \mathcal{R}e, \varphi e = ae$.

Denote by

$$\mu := \frac{\deg}{\mathrm{rk}}.$$

From now on in this subsection, $X \subseteq Y$ means a strictly sub-object, thus

$$0 \to X \to Y \to Y/X \to 0$$

is exact.

Definition 3.15. $X \in \mathsf{C}$ is called *semi-stable* if for any nonzero strictly sub-object $C' \subset X$, $\mu(X') \leq \mu(X)$.

Remark 3.16. Any morphism in C has a kernel and coker. The kernel of $f : X \to Y$ is the schematical closure of $\ker(\mathcal{F}(f))$.

Theorem 3.17. *For any nonzero $X \in \mathsf{C}$, there is a unique filtration*

$$0 = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = X$$

such that X_i/X_{i-1} is semi-simple and

$$\mu(X_1/X_0) > \cdots > \mu(X_n/X_{n-1}).$$

Define the Harder-Narasimhan polygon $\mathrm{HN}(X)$ to be the concave polygon defined on $[0, \mathrm{rk}X]$ with breaking points $(\mathrm{rk}X_i, \deg X_i)$, i.e., on $[\mathrm{rk}X_i, \mathrm{rk}X_{i+1}]$, it has slope $\mu(X_{i+1}/X_i)$.

Theorem 3.18. *For any $Y \subseteq X$, $(\mathrm{rk}Y, \deg Y)$ is under $\mathrm{HN}(X)$. Thus $\mathrm{HN}(X)$ is the concave hull of $(\mathrm{rk}Y, \deg Y)$ for all $Y \subseteq X$.*

Theorem 3.19. *The subcategory $\mathsf{C}_\lambda^{\mathrm{ss}}$ of slope λ semi-simple objects. is an abelian category, stable under extensions in C. Thus the Harder-Narasimhan filtrations give a dévissage of C in $(\mathsf{C}_\lambda^{\mathrm{ss}})_{\lambda\in\mathbb{R}}$.*

Proof of existance. If

$$0 \to X' \to X \to X'' \to 0$$

is exact, then

$$\mu(X) = \frac{\mathrm{rk}X'}{\mathrm{rk}X}\mu(X') + \frac{\mathrm{rk}X''}{\mathrm{rk}X}\mu(X'') \in [\mu(X'), \mu(X'')].$$

Here $[a, b] := [b, a]$ if $a > b$, i.e., the convex hull $\mathrm{Conv}(a, b)$.

If

$$0 = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = X$$

is a Harder-Narasimhan filtration of X, then

$$\mu(X) \in \mathrm{Conv}(\mu(X_i/X_{i-1}))_{1\leq i\leq n}.$$

Thus

$$\inf\{\mu(X_i/X_{i-1}\} \leq \mu(X) \leq \sup\{\mu(X_i/X_{i-1}\}.$$

For nonzero X in C, consider the condition

(*) $\qquad Y \subseteq X$ semi-stable and for any $Y' \subsetneq Y \subset X$, $\mu(Y') \leq \mu(Y)$,

i.e., Y is maximal semi-stable sub-object of X. This is equivalent to say, any nonzero $Y'' \subset X/Y, \mu(Y'') < \mu(Y)$. In fact, if $Y'' = Y'/Y, Y \subsetneq Y' \subset X$, $\mu(Y') \in (\mu(Y), \mu(Y''))$ and thus $\mu(Y'') < \mu(Y)$.

Lemma 3.20. *At most one $Y \subseteq X$ satisfying (*).*

Assume Y_1, Y_2 satisfy (*). Suppose $Y_1 \not\subseteq Y_2$, consider

$$\begin{tikzcd}
\mathrm{Ker}f \arrow[r] & Y_1 \arrow[rr, "f"] \arrow[dr] & & X/Y_2 \\
& & \mathrm{Im}f \arrow[ur, "\subseteq"'] &
\end{tikzcd}$$

$Y_1/\mathrm{Ker}f \to \mathrm{Im}f$ is an isomorphism in generic fibers, thus $\mu(Y_1/\mathrm{Ker}f) \leq \mu(\mathrm{Im}f)$. But Y_1 is semi-stable, $\mu(\mathrm{Ker}f) \leq \mu(Y_1) \leq \mu(Y_1/\mathrm{Ker}f) \leq \mu(\mathrm{Im}f) < \mu(Y_2)$. By symmetric, $\mu(Y_2) < \mu(Y_1)$ if $Y_2 \not\subset Y_1$. Thus $Y_1 \subseteq Y_2$ or $Y_2 \subseteq Y_1$.

Lemma 3.21. $\mu_{\max}(X) := \sup\{\mu(Y) \mid 0 \neq Y \subset X\} < +\infty$.

Take

$$0 = X_0 \subsetneq \cdots \subsetneq X_n = X$$

such that $0 = \mathcal{F}(X_0) \subsetneq \cdots \subsetneq \mathcal{F}(X_n) = \mathcal{F}(X)$ is a Jordan-Hölder filtration. For nonzero $Y \subseteq X$, take $0 = Y_0 \subseteq \cdots \subseteq Y_n = Y$ such that $\mathcal{F}(Y_i) = \mathcal{F}(Y) \cap \mathcal{F}(X_i)$. Consider $u_i : Y_i/Y_{i-1} \hookrightarrow X_i/X_{i-1}$, $\mathcal{F}(u_i) : \mathcal{F}(Y_i/Y_{i-1}) \hookrightarrow \mathcal{F}(X_i/X_{i-1})$. Since $\mathcal{F}(X_i/X_{i-1})$ is simple, $Y_i = Y_{i-1}$ or $\mathcal{F}(u_i)$ is an isomorphism, thus $\mu(Y_i/Y_{i-1}) \leq \mu(X_i/X_{i-1})$ and then $\mu(Y) \leq \sup\{\mu(Y_i/Y_{i-1})\} \leq \sup\{\mu(X_i/X_{i-1})\}$.

Lemma 3.22. $\mu_{\max}(X)$ *is reached.*

It's clear if $\deg : \mathsf{C} \to \mathbb{Z}$.

Now we take Y such that $\mu(Y) = \mu_{\max}(X)$ with maximal rank, then Y satisfies (*).

Let's back to the proof. Set $X_1 \subset X$ satisfying (*) and $X_i/X_{i-1} \subset X/X_{i-1}$ satisfying (*) inductively. The existance then follows.

If we have such a filtration, then $X_1 \subset X$ satisfying (*). In fact, for $Y \subset X/X_1$, $0 = Y_1 \subset Y_2 \subset \cdots \subset Y_n = Y$ such that $v_i : Y_i/Y_{i-1} \hookrightarrow X_i/X_{i-1}$. Then $\mu(Y_i/Y_{i-1}) \leq \mu(\mathrm{Im}v_i) \leq \mu(X_i/X_{i-1})$ and $\mu(Y) \leq \sup\{\mu(Y_i/Y_{i-1})\} \leq \sup\{\mu(X_i/X_{i-1})\} = \mu(X_1/X_0)$. The uniqueness then follows by induction. □

4. Classification of vector bundles

Assume $E/\mathbb{Q}_p$, $F/\mathbb{F}_q$ is algebraically closed. Let $X_E/\mathrm{Spec}E$ be the Fontaine-Fargues curve.

Theorem 4.1 (GAGA, Kedlaya-Liu)**.** *There is an equivalence of categories*

$$\mathsf{Coh}_X \xrightarrow{\sim} \mathsf{Coh}_{X^{\mathrm{an}}}.$$

4.1. **Construction of some vector bundles.** Recall $X_E = \mathrm{Proj}(P_{E,\pi})$. Denote by $\mathcal{O}_{X_E}(d)$ the module with respect to the graded $P_{E,\pi}$-module $P_{E,\pi}[d]$. This is a line bundle on X_E.

Remark 4.2. X_E does not depend canonically on the choice of π, but $\mathcal{O}_{X_E}(1)$ does: another choice of uniformizing element leads to an isomorphic line bundle but the isomorphism is not canonical.

Since X is "complete", $\deg(\mathrm{div}f) = 0$, we have

$$\deg : \mathrm{Pic}(X_E) = \mathrm{Div}(X_E)/\mathrm{div}(E(X_E)^\times) \to \mathbb{Z}.$$

Define $\deg(\mathcal{E}) = \deg(\det \mathcal{E})$ for vector bundle $\mathcal{E}$. Take $\mu = \deg/\mathrm{rk}$, we get Harder-Narasimhan reduction theory.

Proposition 4.3. *We have an isomorphism* $\deg : \mathrm{Pic}(X_E) \xrightarrow{\sim} \mathbb{Z}$, *i.e.* $\mathrm{Pic}(X_E) = \langle \mathcal{O}_{X_E}(1) \rangle$.

This is a consequence of $X_E - \{\infty\}$ is affine and the ring of global sections are PID.

For E'/E, $X'_E := X_E \otimes_E E'$. If E_h/E is unramified of degree h, then $\varphi_{E_h} = \varphi_E^h, W_{\mathcal{O}_{E_h}} = W_{\mathcal{O}_E}$. Replacing E by E_h does not change $Y_{E_h} = Y_E$, it changes the

Frobenius.

$$\begin{tikzcd} X^{\mathrm{ad}}_{E_h} \arrow[r, equal] \arrow[d, "\mathbb{Z}/n\mathbb{Z}"'] & Y^{\mathrm{ad}}/\varphi^{n\mathbb{Z}} \arrow[d, "\pi_h"] \\ X^{\mathrm{ad}}_E \arrow[r, equal] & Y^{\mathrm{ad}}/\varphi^{\mathbb{Z}}. \end{tikzcd}$$

Then by GAGA, we get a $\mathbb{Z}/h\mathbb{Z}$ Galois cover

$$\begin{tikzcd} X_{E_h} \arrow[d, "\pi_h"] \\ X_E. \end{tikzcd}$$

Thus

$$\begin{tikzcd} (X_{E_h})_{h\geq 1} \arrow[d] \\ X_E \end{tikzcd}$$

is a $\widehat{\mathbb{Z}}$-pro Galois cover.

We have $\pi^*_{E_h}\mathcal{O}_{X_E}(d) = \mathcal{O}_{X_{E_h}}(hd)$.

Definition 4.4. For any $\lambda = d/h \in \mathbb{Q}, (d,h) = 1, h > 0$, define

$$\mathcal{O}_{X_E}(\lambda) = \pi_h * \mathcal{O}_{X_{E_h}}(d).$$

It's of rank h and degree d. It's semi-stable of slope λ since pushforwards of a semi-stable vector bundle by a finite étale Galois cover are still semi-stable.

We have

$$\mathcal{O}(\lambda) \otimes \mathcal{O}(\mu) = \bigoplus_{\text{finite}} \mathcal{O}(\lambda + \mu),$$

$$\mathcal{O}(\lambda)^\vee = \mathcal{O}(-\lambda).$$

$$\mathrm{Hom}(\mathcal{O}(\lambda), \mathcal{O}(\mu)) = \bigoplus_{\text{finite}} \mathrm{H}^0(X, \mathcal{O}(\mu - \lambda)$$

is zero if $\lambda > \mu$ since $\mathrm{H}^0(X_E, \mathcal{O}(\frac{d}{h})) = \mathrm{H}^0(X_{E_h}, \mathcal{O}_{X_{E_h}}(d)) = 0$ if $d < 0$.

$$\mathrm{Ext}^1(\mathcal{O}(\lambda), \mathcal{O}(\mu)) = \bigoplus_{\text{finite}} \mathrm{H}^1(X, \mathcal{O}(\mu - \lambda))$$

is zero if $\lambda \leq \mu$ since $\mathrm{H}^1(X_E, \mathcal{O}(\frac{d}{h})) = \mathrm{H}^1(X_{E_h}, \mathcal{O}_{X_{E_h}}(d)) = 0$ if $d \geq 0$.

Theorem 4.5. *(1) Any slope λ semi-stable vector bundle is isomorphic to a direct sum of $\mathcal{O}_X(\lambda)$.*

(2) The Harder-Narasimhan filtration of a vector bundle is split.

(3) There is a bijection between

$$\{\lambda_1 \geq \cdots \geq \lambda_n \mid \lambda_i \in \mathbb{Q}, n \in \mathbb{N}\}$$

and the isomorphic classes of vector bundles on X as

$$(\lambda_i) \longmapsto \left[\bigoplus_i \mathcal{O}(\lambda_i)\right].$$

Remark 4.6. (1)+(2) $\Longleftrightarrow$ (3). Moreover, (1)$\Rightarrow$(2) via the computation of $\mathrm{Ext}^1(\mathcal{O}(\lambda), \mathcal{O}(\mu)) = 0$ if $\lambda \leq \mu$.

In particular, denote by $\mathsf{Bun}_X^{,0}$ the abelian category of slope 0 semi-stable vector bundles over X. Then we have an equivalence of categories

$$\begin{aligned} \mathsf{Vect}_E &\xrightarrow{\sim} \mathrm{Bun}_X^{,0} \\ V &\mapsto V \otimes_E \mathcal{O}_X \\ \mathrm{H}^0(X, \mathcal{E}) &\leftarrow\!\shortmid \mathcal{E}. \end{aligned}$$

That's to say, a vector bundle over X is trivial iff it's semo-stable of slope 0.

More generally, $\mathrm{End}(\mathcal{O}(\lambda)) = D_\lambda^{\mathrm{op}}$, where D_λ is the division algebra over E with invariant λ. We have an equivalence of categories

$$\begin{aligned} \mathsf{Vect}_{D_\lambda} &\xrightarrow{\sim} \mathrm{Bun}_X^{,\lambda} \\ V &\mapsto V \otimes_{D_\lambda} \mathcal{O}(\lambda) \end{aligned}$$

4.2. **From isocrystals to vector bundles.** Denote by $\breve{E} = \widehat{E^{\mathrm{ur}}}$ endowed with Frobenius σ. Denote by $\varphi\text{-}\mathsf{Mod}_{\breve{E}}$ the abelian category of isocrystals, which is semi-stable by Dieudonné-Mannin.

$$\varphi\text{-}\mathsf{Mod}_{\breve{E}} = \bigoplus_{\lambda \in \mathbb{Q}} \varphi\text{-}\mathsf{Mod}_{\breve{E}}^{\lambda}.$$

For any λ, there is a unique simple object $N_\lambda = \langle e, \varphi(e), \dots, \varphi^{h-1}(e) \rangle, \lambda = d/h$ with $\varpi^h(e) = \pi^d e$.

We have a $\otimes$-exact functor

$$\begin{aligned} \varphi\text{-}\mathsf{Mod}_{\breve{E}} &\longrightarrow \mathsf{Bun}_X \\ (D, \varphi) &\longmapsto \mathcal{E}(D, \varphi) \end{aligned}$$

where $\mathcal{E}(D, \varphi)$ is the module associated to the graded P-module

$$\bigoplus_{d \geq 0} (D \otimes_E B)^{\varphi \otimes \varphi = \pi^d}.$$

Via GAGA, $\mathcal{E}(D, \varphi)^{\mathrm{ad}}$ is a vector bundle on $Y/\varphi^{\mathbb{Z}}$ corresponding to the φ-equivariant vector bundle $(D \otimes_{\breve{E}} \mathcal{O}_Y, \varphi \otimes \varphi)$.

If (D, φ) is simple of slope λ, then $\mathcal{E}(D, \varphi) = \mathcal{O}_X(-\lambda)$. Thus via Dieudonné-Manin classification theorem, this functor is essentially surjective.

5. Periods of p-divisible groups

The main tool is the classification theorem. Take $E = \mathbb{Q}_p$ to simplify. Let $C/\mathbb{Q}_p$ be an algebraically closed field with $C^\flat = F$. Thus there is $\infty \in |X|$ with $k(\infty) = C$.

Denote by $\mathsf{BT}_{\mathcal{O}_C}$ the category of Barsotti-Tate p-divisible groups over $\mathcal{O}_C$. We want to explain the functor

$$\begin{aligned} \mathsf{BT}_{\mathcal{O}_C} &\longrightarrow \{\text{Modifications of vector bundles}\} \\ H &\longmapsto [0 \to V_p(G) \otimes \mathcal{O}_X \to \mathcal{E}_H \to i_{\infty *}\mathrm{Lie} H[\frac{1}{p}] \to 0] \end{aligned}$$

where $V_p(H) \otimes \mathcal{O}_X$ is a trivial vector bundle with fiber $V_p(H)$, $\mathcal{E}_H = \mathcal{E}(D, p^{-1}\varphi)$ is a covariant isocrystal of the reduction of H.

5.1. **Periods in characteristic p.** Let $k/\mathbb{F}_p$ be a perfect field. A Dieudonné crystal is a free $W(k)$-module of finite rank with endomorphisms F, V, where F is σ-linear, V is σ^{-1}-linear, $FV = VF = p$. Then

$$\begin{aligned} \mathsf{BT}_k &\xrightarrow{\sim} \{\text{Dieudonné crystals}\} \\ H &\mapsto \mathbb{D}(H). \end{aligned}$$

5.2. **The covectors.** Denote by

$$W_n = \{[x_0, \dots, x_{n-1}]\} = W/V^n W$$

the ring of trucated Witt vectors of length n. It's an affinte unipotent group scheme, isomorphic to $\mathbb{A}^n_k$. We have

$$W_n \xhookrightarrow{V} W_{n+1} \xhookrightarrow{V} W_{n+2} \hookrightarrow \cdots$$

where $V([x_0, \dots, x_{n-1}]) = [0, x_0, \dots, x_{n-1}]$.

Denote by

$$\mathrm{CW}^u := \varinjlim_{n\geq 1} W_n = \{[x_n]_{n\leq 0} \mid x_n = 0 \text{ for } n \ll 0\}.$$

the ring of unipotent Witt covectors. Here

$$[x_n] + [y_n] = [z_n]$$

with $x_n = P_k(x_{n-k}, \dots, x_n, y_{n-k}, \dots, y_n), k \gg 0$, P_k is the polynomial gives the addition of Witt vectors

$$\sum_{n\geq 0} V^n[x_n] + \sum_{n\geq 0} V^n[y_n] = \sum_{n\geq 0} V^n[P_n(x_0, \dots, x_n, y_0, \dots, y_n)].$$

The problem of this ring is $\mathrm{Hom}(\mu_p, \mathrm{CW}^u) = 0$ since μ_p is not unipotent. So we need Fontaine's Witt covectors. Let R be an $\mathbb{F}_p$-algebra,

$$\mathrm{CW}(R) := \{[x_n] \mid x_n \in R, (x_n)_{n\leq N} \text{ nilpotent } N \ll 0\}.$$

It's well-define, i.e., for any n, the sequence

$$(P_k(x_{n-k,\dots,x_n,y_{n-k},\dots,y_n}))_{k\geq 0}$$

is constant for $k \gg 0$.

We have $F[x_n] = [x_n^p], V[\dots, x_{-1}, x_0] = [\dots, x_{-2}, x_{-1}]$. For $H \in \mathsf{BT}_k$,

$$\mathbb{D}(H) = \mathrm{Hom}_k(H, \mathrm{CW}_k).$$

It's some kind of Pontryagin duality. The action of F, V via them on CW. Then if $M = \mathbb{D}(H)$, one finds back H via

$$H = \mathrm{Hom}_{F,V}(M, \mathrm{CW}).$$

Example 5.1. $M = W(k) \cdot e$, $Fe = e, Ve = pe$, R is an $\mathbb{F}_p$-algebra.

$$\mathrm{Hom}_{F,V}(M, \mathrm{CW}(R)) = \left\{ [x_n]_{n\leq 0} \mid x_n \in R, x_n^p = x_n, \sum_{n\leq N} Rx_n \text{ nilpotent } , N \ll 0 \right\}.$$

Thus $x_n = 0$ for $n \ll 0$ and

$$\mathrm{Hom}_{F,V}(M, \mathrm{CW}(R)) = \mathbb{Q}_p/\mathbb{Z}_p(R).$$

This means $M = \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)$, $\mathbb{Q}_p/\mathbb{Z}_p = \{[x_n]_{n\leq 0} \in \mathrm{CW} \mid x_n^p = x_n\}$.

Example 5.2. $M = W(k) \cdot e, Fe = pe, Ve = e,$

$$\mathrm{Hom}_{F,V}(M, \mathrm{CW}(R)) = \{[x_n]_{n\leq 0} \mid x_n \in R, x_{n-1} = x_n, x_n \text{ nilpotent }\} = \widehat{\mathbb{G}}_m(R).$$

Then $M = \mathbb{D}(\widehat{\mathbb{G}}_m)$, $\widehat{\mathbb{G}}_m \xrightarrow{\sim} \mathrm{CW}^{V=\mathrm{Id}}, x \mapsto \sum_{n\leq 0} V^n[x]$.

Example 5.3. Let $\lambda = d/h \in (0,1), d \geq 1, (d,h) = 1$. Denote

$$\begin{aligned} H_\lambda &= \mathrm{Ker}(V^d - F^{h-d} : \mathrm{CW} \to \mathrm{CW}) \\ &= \left\{ [\dots, z_{d-1}^{p^{h-d}}, \dots, z_1^{p^{h-d}}, z_{d-1}, \dots, z_1] \in \mathrm{CW} \mid z_1, \dots, z_{d-1} \text{ nilpotent} \right\} \end{aligned}$$

the formal p-divisible group of slope λ. Then $H_\lambda = \mathrm{Spf}(k[[z_0, \dots, z_{d-1}]])$. Denote by $M_\lambda = \mathbb{D}(H_\lambda)$. Then $(M_\lambda[\frac{1}{p}], F)$ is a simple isocrystal of slope λ.

If $[x_k]_{k\geq 0}+[y_k]_{k\geq 0}=[p_k(x_0,\dots,x_k,y_0,\dots,y_k)]_{k\geq 0}$, then

$$(x_0,\dots,x_{-d+1})+_{H_\lambda}(y_0,\dots,y_{-d+1})=(z_0,\dots,z_{-d+1}),$$

$$z_0=\lim_{k\to+\infty}P_{kd}(x_{d-1}^{p^{k(h-d)}},\dots,x_0^{p^{k(h-d)}},\dots,x_{-d+1},\dots,x_0,\dots,y_0)$$

for the $(x_0,\dots,x_{-d+1},y_0,\dots)$-adic topology on $k[[x_i,y_i]]$.

5.3. **Period isomorphism in characteristic** p. Let $F/\overline{\mathbb{F}}_p$ be a perfectoid field, H a p-divisible formal group over $\overline{\mathbb{F}}_p$. Let $M=\mathbb{D}(H)$ be the contravariant Dieudonné module. Denote

$$\mathrm{BW}=\varprojlim_V \mathrm{CW}=\{[x_n]_{n\in\mathbb{Z}}\mid (x_n)_{n\leq N}\text{ is nilpotent}, N\ll 0\}.$$

Then

$$0\to W\to \mathrm{BW}\to\mathrm{CW}\to 0$$

is exact.

Since

$$H(\mathcal{O}_F)=\mathrm{Hom}(\mathrm{Spf}\mathcal{O}_F,H)=\varprojlim_{(0)\neq\mathfrak{a}\subset\mathcal{O}_F}H(\mathcal{O}_F/\mathfrak{a}),$$

$$\mathrm{CW}(\mathcal{O}_F)=\varprojlim\mathrm{CW}(\mathcal{O}_F/\mathfrak{a})=\left\{[x_n]_{n\leq 0}\mid x\in\mathcal{O}_F,\limsup_{n\to-\infty}|x_n|<1\right\}.$$

We have

$$H(\mathcal{O}_F)=\mathrm{Hom}_{W(k)[F,V]}(M,\mathrm{CW}(\mathcal{O}_F)),$$

H is formal if and only if F is topologically nilpotent on M and $\mathcal{O}_F$ is perfect.

Proposition 5.4. *The projection* $\mathrm{BW}(\mathcal{O}_F)\twoheadrightarrow\mathrm{CW}(\mathcal{O}_F)$ *induces*

$$\mathrm{Hom}_{W(k)[F,V]}(M,\mathrm{BW}(\mathcal{O}_F))\xrightarrow{\sim}\mathrm{Hom}_{W(k)[F,V]}(M,\mathrm{CW}(\mathcal{O}_F)).$$

An inverse is given by

$$u\mapsto[x\mapsto\lim_{k\to+\infty}F^{-k}\widetilde{u(F^kx)}].$$

If $(D,\varphi)=(M[\frac{1}{p}],F)$, one deduces

$$H(\mathcal{O}_F)=\mathrm{Hom}_\varphi(D,\mathrm{BW}(\mathcal{O}_F)).$$

Now

$$\mathrm{BW}(\mathcal{O}_F)\hookrightarrow\mathcal{O}(Y_F)=B_F,$$
$$V^n[x_n]\mapsto[x_n^{p^{-n}}]p^n.$$

Thus

$$\mathrm{BW}=\left\{\sum_{n\in\mathbb{Z}}[x_n]p^n\mid x_n\in\mathcal{O}_F,\limsup_{n\to-\infty}|x_n|^{p^n}<1\right\}\subset B_F^+=\mathcal{O}(Y_F\cup\{y_{\mathrm{cris}}\})$$

contains all periods with slope in $[0,1]$.

Proposition 5.5. $\mathrm{Hom}_\varphi(D,\mathrm{BW}(\mathcal{O}_F))=\mathrm{Hom}_\varphi(D,B_F)$.

Example 5.6. For $\lambda=d/h\in(0,1]$,

$$\begin{aligned}H_\lambda(\mathcal{O}_F)&=B_F^{\varphi^h=p^d}=\mathrm{BW}(\mathcal{O}_F)^{V^d=F^{h-d}}\\&=\left\{\sum_{k=0}^{d-1}\sum_{n\in\mathbb{Z}}[x_k^{p^{-nh}}]p^{nd+k}\mid x_0,\dots,x_{d-1}\in\mathfrak{m}_F\right\}.\end{aligned}$$

If $\lambda = 1$, we have an isomorphism

$$\begin{aligned} \mathfrak{m}_F &\xrightarrow{\sim} B^{\varphi=p} \\ \varepsilon &\mapsto \sum_{n\in\mathbb{Z}} [\varepsilon^{p^{-n}}]p^n. \end{aligned}$$

Denote by

$$\mathcal{L} = \sum_{n\geq 0} \frac{T^{p^n}}{p^n} \in \mathbb{Q}_p[[T]]$$

the logarithm of a p-typical formal group law $\mathcal{F}/\mathbb{Z}_p$. Then

$$X +_{\mathcal{F}} Y = \mathcal{L}^{-1}(\mathcal{L}(X) + \mathcal{L}(Y)) \in \mathbb{Z}_p[[X,Y]].$$

For $X +_{\widehat{\mathbb{G}}_m} Y = XY + X + Y, \log_{\widehat{\mathbb{G}}_m} = \log(1+T)$. Denote by $E(T) = \exp(\mathcal{L}(T)) \in \mathbb{Z}_p[[T]]$ the Artin-Hasse map. Then $E : \mathcal{F} \xrightarrow{\sim} \widehat{\mathbb{G}}_m$ and we have a commutative diagram

$$\begin{tikzcd} (\mathfrak{m}_F, +_{\mathcal{F}}) \arrow[rr, "\sim", "\varepsilon\mapsto \sum_{n\in\mathbb{Z}}[\varepsilon^{p^{-n}}]p^n"'] \arrow[d, "E"', "\simeq"] & & B^{\varphi=p} \arrow[d, equal] \\ (\mathfrak{m}_F, +_{\widehat{\mathbb{G}}_m}) \arrow[rr, "\sim", "\varepsilon\mapsto\log([1+\varepsilon])=t"'] & & B^{\varphi=p}. \end{tikzcd}$$

If $\lambda = d/h \notin [0,1]$, $B^{\varphi^h=p^d}$ has no explicit description: the Banach-Colmez space $\mathbb{B}^{\varphi^h=p^d}$ is not representatble by a pefectoid space but by a diamond (algebraic space for pro-étale topology).

5.4. **Periods in unequal characteristic.** Let $C/\mathbb{Q}_p$ be an algebraically closed field, $F = C^\flat$, $H/\mathcal{O}_C$ a formal p-divisible group. We are going to look at the universal cover $\varprojlim_{\times p} H$ of H.

Proposition 5.7. *There is an isomorphism* $\varprojlim_{\times p} H(\mathcal{O}_C) \xrightarrow{\sim} \varprojlim_{\times p} H(\mathcal{O}_C/p\mathcal{O}_C)$. *The inverse is given by sending* $(x_n)_{n\geq 0}$ *to* $(\lim_{k\to+\infty} p^{-k}\widetilde{x}_{n+k})_{n\geq 0}$ *via any lift of* $H(\mathcal{O}_C) = \varprojlim_{\times p} H(\mathcal{O}_C/p^i\mathcal{O}_C) \to H(\mathcal{O}_C/p\mathcal{O}_C)$.

The last isomorphism comes from that H is p-divisible p^∞-torsion, $H_\eta = \overset{\circ}{B}{}^d_C$, while $\times p$ contracts everything to 0.

Suppose $\mathbb{H}/\overline{\mathbb{F}}_p$ is a p-divisible group with an identification

$$\mathbb{H} \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_C/p\mathcal{O}_C \xrightarrow{\sim} H \otimes_{\mathcal{O}_C} \mathcal{O}_C/p\mathcal{O}_C.$$

Take $\varpi^\sharp = p$, then

$$\begin{aligned} \varprojlim_{\times p} H(\mathcal{O}_C) = \varprojlim_{\times p} H(\mathcal{O}_C/p\mathcal{O}_C) &= \varprojlim_{\times p} \mathbb{H}(\mathcal{O}_C/p\mathcal{O}_C) = \varprojlim_{\times p} \mathbb{H}(\mathcal{O}_F/\varpi\mathcal{O}_F) \\ &= \varprojlim_{\times p} \mathbb{H}(\mathcal{O}_F) = \mathbb{H}(\mathcal{O}_F) = \mathrm{Hom}_\varphi(D, B_F), \end{aligned}$$

where $(D,\varphi) = \mathbb{D}(\mathbb{H})$.

Remark 5.8. More generally

$$\varprojlim_{\times p} H_\eta = \overset{\circ}{B}{}^{d,1/p^\infty}_C$$

is a pre-perfectoid ball $\mathrm{Spf}[[X_0^{1/p^\infty}, \dots, X_{d-1}^{1/p^\infty}]]_\eta$ over C, where $H_\eta = \overset{\circ}{B}{}^d_C$. The tilt of this is $(\mathbb{H}^{1/p^\infty} \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_F)_\eta$.

Let

$$\log_H : H_\eta \to \mathrm{Lie} H \otimes_{\mathcal{O}_C} \mathbb{G}_a^{\mathrm{rig}}$$

be the logarithm of the formal group H_η. This is a morphism of rigid analytic groups, which is an étale $H(\mathcal{O}_C)[p^\infty]$-tower.

By applying $\varprojlim_{\times p}$ on the exact sequence

$$0 \to H(\mathcal{O}_C)[p^\infty] \to H_\eta \xrightarrow{\log_H} \mathrm{Lie} H \otimes_{\mathcal{O}_C} \mathbb{G}_a^{\mathrm{rig}} \to 0,$$

we get

$$0 \to V_p(H) \to \varprojlim_{\times p} H(\mathcal{O}_C) \xrightarrow{\log_H(x_0)} \mathrm{Lie} H[\frac{1}{p}] \to 0.$$

Rewrite it in terms of covariant isocrystals, we get

$$0 \to V_p(H) \to (D \otimes_{\breve{\mathbb{Q}}_p} B_F)^{\varphi = p} \to \mathrm{Lie} H[\frac{1}{p}] \to 0.$$

Here let $\mathrm{Fil} D_C = \omega_{H^D}[\frac{1}{p}] \subset D_C$ be the Hodge filtration. Then $D_C / \mathrm{Fil} D_C = \mathrm{Lie} H[\frac{1}{p}]$ and the last map in the exact sequence is given by

$$\begin{tikzcd}
(D \otimes_{\mathbb{Q}_p} B_F)^{\varphi = \mathrm{Id}} \arrow[r] \arrow[d, hook] & \mathrm{Lie} H[\frac{1}{p}] \\
D \otimes_{\breve{\mathbb{Q}}_p} B_F \arrow[r, two heads, "\mathrm{id} \otimes \theta"] & D_C \arrow[u, two heads]
\end{tikzcd}$$

Example 5.9. When $H = \widehat{\mathbb{G}}_m$, this is just the fundamental exact sequence.

Proposition 5.10. $V_p(H) \to (D \otimes B)^{\varphi = p}$ *induces an isomorphism*

$$V_p(H) \otimes_{\mathbb{Q}_p} B[\frac{1}{p}]^{\varphi = \mathrm{Id}} \xrightarrow{\sim} (D \otimes_{\breve{\mathbb{Q}}_p} B[\frac{1}{t}])^{\varphi = \mathrm{Id}}.$$

Use Poincaré duality, we get a perfect pairing

$$\begin{tikzcd}
V_p(H) \times V_p(H^D) \arrow[r, "\cup"] \arrow[d] & \mathbb{Q}_p(1) = \mathbb{Q}_p t \arrow[d, hook] \\
(D \otimes B)^{\varphi = p} \times (D^* \otimes B)^{\varphi = \mathrm{Id}} \arrow[r, "\cup"] & B^{\varphi = p}.
\end{tikzcd}$$

The right hand side map is an isomorphism after inverting t.

Corollary 5.11. *For any p-divisible group $H/\mathcal{O}_C$, the corresponding $(D, \varphi, \mathrm{Fil} D_C)$ defines a modification of vector bundles on X_F at $\infty \in |X_F|$,*

$$0 \to V_p(H) \otimes_{\mathbb{Q}_p} \mathcal{O}_X \to \mathcal{E}(D, p^{-1}\varphi) \to i_{\infty *} \mathrm{Lie} H[\frac{1}{p}] \to 0.$$

In particular, via $D_C = \mathcal{E}(D, p^{-1}\varphi)_\infty \otimes k(\infty)$, $u : \mathcal{E}(D, p^{-1}\varphi) \twoheadrightarrow i_{\infty *} D_C$, $u^{-1}(i_{\infty *} \mathrm{Fil} D_C)$ is a trivial bundle.

6. Topics on classification theorem

6.1. **Lubin-Tate space.** Let $\mathbb{H}$ be a 1-dimensional hegith n formal p-divisible group. Let

$$\mathfrak{X} = \mathrm{Def}(\mathbb{H}) \simeq \mathrm{Spf}(W(\overline{\mathbb{F}}_p)[[X_1, \dots, X_{n-1}]]).$$

Then we have Gross-Hopkins period morphism, which is an anlog of Griffiths period morphism.

$$\begin{array}{c} \mathfrak{X}_\eta = \overset{\circ}{B}{}^{n-1}_{\breve{\mathbb{Q}}_p} \\ \big\downarrow{\scriptstyle \pi_{\mathrm{dR}}} \\ \mathbb{P}^{n-1}_{\breve{\mathbb{Q}}_p} \end{array}$$

Denote $(D,\varphi)=\mathbb{D}(\mathbb{H})$. Then for $x\in\mathfrak{X}(\mathcal{O}_C)=\mathfrak{X}_\eta(C)$, $\pi_{\mathrm{dR}}(x)=\mathrm{Fil}D_C\subset D_C$ is a codimension $1=\dim\mathbb{H}$ subspace, that is, the Hodge filtration of $x^*H^{\mathrm{univ}}/\mathcal{O}_C$, where $H^{\mathrm{univ}}/\mathfrak{X}$ is a universal deformation.

Theorem 6.1 (Lafaille, Gross-Hopkins). *π_{dR} is a surjective étale cover.*

That's to say, any codimension one subspace $\mathrm{Fil}D_C$ is the Hodge filtration of a lift of $\mathbb{H}$ to $\mathcal{O}_C$. This is a p-adic analog of Kodaira-Spencer map. The étaleness follows from Grothendieck-Messing deformation theory.

We have $\mathcal{E}(D,p^{-1}\varphi)=\mathcal{O}_X(\frac{1}{n})$.

Corollary 6.2. *For any degree 1 modification of $\mathcal{O}_X(\frac{1}{n})$,*

$$0\to\mathcal{E}\to\mathcal{O}_X(\frac{1}{n})\to\mathcal{F}\to 0$$

where $\mathcal{F}$ is a degree 1 torsion coherent sheaf, we have trivial $\mathcal{E}\simeq\mathcal{O}_X^n$.

Conversely,

Proposition 6.3. *For*

$$0\to\mathcal{O}_X^n\to\mathcal{E}\to\mathcal{F}\to 0$$

where $\mathcal{F}$ is a degree 1 torsion coherent sheaf, we have $\mathcal{E}=\mathcal{O}_X(\frac{1}{d})\oplus\mathcal{O}_X^{n-d}, 1\le d\le n$.

The modification is given by a surjection

$$u: C(-1)^n=(t^{-1}B^+_{\mathrm{dR}}/B^+_{\mathrm{dR}})^n\twoheadrightarrow L,$$

where L is a one-dimensional C-vector space. Here $\widehat{\mathcal{O}}^n_{X,\infty}\subset\widehat{\mathcal{E}}_\infty\subset t^{-1}\widehat{\mathcal{O}}^n_{X,\infty}$. Up to replacing $\mathcal{O}_X^n$ by $\mathcal{O}_X^{n-i}$ and $\mathcal{E}$ by $\mathcal{E}'$ with $\mathcal{E}=\mathcal{E}'\oplus\mathcal{O}_X^i$, one can suppose $u:\mathbb{Q}_p(-1)^n\hookrightarrow L$, i.e., $u\in\Omega(C)\subset\mathbb{P}^{n-1}(C)$.

We want to prove this if $u\in\Omega(C)$, then $\mathcal{E}\cong\mathcal{O}_X(\frac{1}{n})$. Let $D=\mathrm{End}(\mathcal{O}_X(\frac{1}{n}))=D_{\frac{1}{n}}$ be the division algebra with invariant $\frac{1}{n}$. It induces $D\otimes_{\mathbb{Q}_p}\mathcal{O}_X\xrightarrow{\sim}\underline{\mathrm{End}}(\mathcal{O}_X(\frac{1}{n}))$ and $D_X^{\mathrm{op},\times}\xrightarrow{\sim}\underline{\mathrm{Aut}}(\mathcal{O}_X(-\frac{1}{n}))=\underline{\mathrm{GL}}(\mathcal{O}_X(-\frac{1}{n}))$ as X-group schemes. Thus $(D^{\mathrm{op}})^\times_X$-torsors over X (pure inner form of GL_n) is equivalent to GL_n-torsors on X (vector bundle of rank n). In fact, if $\mathcal{T}$ is a topos, G is a group on $\mathcal{T}$, $\mathbb{T}$ is a G-torsor in $\mathcal{T}$, $H=G^{\mathbb{T}}$ is the inner twisting of G,

$$[\mathbb{T}]\in\mathrm{H}^1(\mathcal{T},G)\to\mathrm{H}^1(\mathcal{T},G_{\mathrm{ad}})\ni[H]=[\underline{\mathrm{Aut}}(\mathbb{T})].$$

Then $t\mapsto\underline{\mathrm{Isom}}(\mathbb{T},t)$ induces the equivalence between G-torsors and H-torsors.

Now

$$0\to\mathcal{O}_X^n\to\mathcal{E}\to\mathcal{F}\to 0$$

is equivalent to

$$0\to\mathcal{O}_X(-\frac{1}{n})\to\mathcal{E}'\to\mathcal{F}'\to 0$$

as $D^{\mathrm{op}}\otimes\mathcal{O}_X$-module. Take dual modification, we get

$$0\to\mathcal{E}''\to\mathcal{O}_X(\frac{1}{n})\to\mathcal{F}''\to 0$$

as $D\otimes\mathcal{O}_X$-module.

Theorem 6.4 (Drinfeld). *Any element of $\Omega(C)$ is the Hodge filtration of a special formal $\mathcal{O}_D$-module.*

Hence $\mathcal{E}'' \simeq D \otimes_{\mathbb{Q}_p} \mathcal{O}_X$. The result follow by applying $\mathrm{Hom}(\mathcal{O}_X(\frac{1}{n}), -)$.

6.2. **Proof of the classification for rank two vector bundles.**

Proposition 6.5. *Let $\mathcal{F}$ be a degree one torsion coherent sheaf on X.*

(1) If

$$0 \to \mathcal{E} \to \mathcal{O}(d_1) \oplus \mathcal{O}(d_2) \to \mathcal{F} \to 0$$

with $d_1 \neq d_2$, $\mathcal{E} \cong \mathcal{O}(d_1 - 1) \oplus \mathcal{O}(d_2)$ or $\mathcal{O}(d_1) \oplus \mathcal{O}(d_2 - 1)$.

(2) If

$$0 \to \mathcal{E} \to \mathcal{O}(d) \oplus \mathcal{O}(d) \to \mathcal{F} \to 0,$$

$\mathcal{E} \cong \mathcal{O}(d - \frac{1}{2})$ or $\mathcal{O}(d-1) \oplus \mathcal{O}(d)$.

(3) If

$$0 \to \mathcal{E} \to \mathcal{O}(d + \frac{1}{2}) \to \mathcal{F} \to 0,$$

$\mathcal{E} \cong \mathcal{O}(d)^2$.

(1) by explicit computation. (2) is a consequence of Lubin-Tate case. (3) is a consequence of Drinfeld case.

Let $\mathcal{E}$ be a rank 2 vector bundle on X. Then there is

$$0 \to \mathcal{O}(d_1) \to \mathcal{E} \to \mathcal{O}(d_2) \to 0.$$

If $d_2 \leq d_1$, $\mathrm{Ext}^1(\mathcal{O}(d_2), \mathcal{O}(d_1)) = 0$ and $\mathcal{E} = \mathcal{O}(d_1) \oplus \mathcal{O}(d_2)$. If $d_2 > d_1$,

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(d_1) \arrow[r] \arrow[d, hook] & \mathcal{E} \arrow[r] \arrow[d, hook] & \mathcal{O}(d_2) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}(d_2) \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{O}(d_2) \arrow[r] & 0.
\end{tikzcd}$$

In both cases,

$$0 \to \mathcal{E} \to \mathcal{O}_X(d)^2 \to \mathcal{F} \to 0.$$

Let $\mathrm{Fil}^\bullet$ be a filtration of $\mathcal{F}$ such that $\mathrm{gr}^i\mathcal{F}$ is zero or degree one torsion coherent sheaf, $\forall i$. Take $\mathrm{Fil}^\bullet\mathcal{O}_X(d)^2 = u^{-1}(\mathrm{Fil}^\bullet\mathcal{F})$. Then for any i, $\mathrm{Fil}^{i+1}(\mathcal{O}_X(d)^2)$ is $\mathrm{Fil}^i(\mathcal{O}_X(d)^2)$, or a degree one modification of $\mathrm{Fil}^i(\mathcal{O}_X(d)^2)$. By induction on $i \in \mathbb{Z}$, we get $\mathrm{Fil}^i(\mathcal{O}_X(d)^2) = \mathcal{O}(k + \frac{1}{2})$ or $\mathcal{O}(k_1) \oplus \mathcal{O}(k_2)$.

6.3. **Weakly admissible implies admissible.** Let $K/\mathbb{Q}_p$ be a discrete valuation field with perfect residue field. Denote $C = \widehat{\overline{K}}, G_K = \mathrm{Gal}(\overline{K}/K)$, $K_0 = W(k_K)_{\mathbb{Q}}$, σ the Frobenius on K_0. Denote by $\varphi\text{-}\mathsf{ModFil}_{K/K_0}$ the category of triples $(D, \varphi, \mathrm{Fil}^\bullet D_K)$, where (D, φ) is an isocrystal and $\mathrm{Fil}^\bullet$ is a Hodge filtration of D_K. Define

$$t_N = v_p(\det \varphi)$$
$$t_H = \sum i \dim \mathrm{gr}^i D_K.$$

Denote

$$\mathbb{V}_{\mathrm{cris}}(D, \varphi, \mathrm{Fil}^\bullet D_K) = \mathrm{Fil}^0(D \otimes_{K_0} B_{\mathrm{cris}})^{\varphi = \mathrm{Id}} = \mathrm{Fil}^0(D \otimes_{K_0} B[\frac{1}{t}])^{\varphi = \mathrm{Id}}.$$

There is a G_K-action on it.

Definition 6.6. $(D, \varphi, \mathrm{Fil}^\bullet D_K)$ is *admissible* if

$$\dim_{\mathbb{Q}_p} \mathbb{V}_{\mathrm{cris}}(D, \varphi, \mathrm{Fil}^\bullet D_K) = \dim_{K_0} D.$$

Definition 6.7. $(D, \varphi, \mathrm{Fil}^\bullet D_K)$ is *weakly admissible* if $t_H = t_N$, and for any sub-isocrystal $D' \subset D$, $t_H(D', \varphi|_{D'}, D'_K \cap \mathrm{Fil}^\bullet D_K) \leq t_N(D', \varphi|_{D'}, D'_K \cap \mathrm{Fil}^\bullet D_K)$.

Theorem 6.8 (Colmez-Fontaine)**.** *Weakly admissible is equivalent to admissible.*

$\Leftarrow$ is easy.

We reinterpretate in terms of semi-stablity. Take $\deg = t_H - t_N$, $\mathrm{rk} = \dim_{K_0} D$, $\mu = \deg / rk$, then $\varphi\text{-}\mathsf{ModFil}^{\mathrm{wa}}_{K/K_0} = \varphi\text{-}\mathsf{ModFil}^{\mathrm{ss},0}_{K/K_0}$.

The action on G_K on $X_{C^\flat}$ stablizes ∞. For any $(D, \varphi, \mathrm{Fil}^\bullet D_K)$, $\mathcal{E}(D, \varphi)$ is a G_K-equivariant vector bundle on X and $\Lambda = \mathrm{Fil}^0(D \otimes B_{\mathrm{dR}})$ is a lattice in $\widehat{\mathcal{E}}_\infty[\frac{1}{t}]$. This gives a modification of $\mathcal{E}$, denoted by $\mathcal{E}(D, \varphi, \mathrm{Fil}^\bullet D_K)$. Then

$$\begin{aligned} & \deg \mathcal{E}(D, \varphi, \mathrm{Fil}^\bullet D_K) \\ = & \deg \mathcal{E}(D, \varphi) + [\mathrm{Fil}^0 D \otimes B_{\mathrm{dR}} : D \otimes B^+_{\mathrm{dR}}] - t_N(D, \varphi) \\ = & \deg(D, \varphi, \mathrm{Fil}^\bullet D_K), \end{aligned}$$

and $\mathrm{H}^0(X, \mathcal{E}(D, \varphi, \mathrm{Fil}^\bullet D_K)) = \mathbb{V}_{\mathrm{cris}}(D, \varphi, \mathrm{Fil}^\bullet D_K)$.

The classification theorem tells that, if $\mathcal{E}$ is a semi-stable vector bundle of slope 0, then $\dim_{\mathbb{Q}_p} \mathrm{H}^0(X, \mathcal{E}) = \mathrm{rk}\mathcal{E}$. Now for $A \in \varphi\text{-}\mathsf{ModFil}_{K/K_0}$,

- A is admissible $\Longleftrightarrow$ $\mathcal{E}(A)$ is semi-stable of slope 0 and for any sub-bundle $\mathcal{E}' \subset \mathcal{E}(A), \mu(\mathcal{E}') \leq 0$;
- A is weakly admissible $\Longleftrightarrow$ A is semi-stable of slope 0 and for any strict sub-object $B \subset A, \mu(B) \leq 0$.

Proposition 6.9. *There is an equivalence between the category of strict subobject of A and G_K-equivariant subobject of $\mathcal{E}(A)$.*

If A is weakly admissible, the Harder-Narasimhan filtration of $\mathcal{E}(A)$ is G_K-invariant. Thus it comes from a filtration of A. Since A is semi-stable, this is the tautological filtration and then $\mathcal{E}(A)$ is semi-stable, A is admissible.